ON THE CLOSING LEMMA FOR PLANAR PIECEWISE SMOOTH VECTOR FIELDS

TIAGO DE CARVALHO[1]

Abstract. A large number of papers deal with "Closing Lemmas" for C^r-vector fields (and C^r-diffeomorphisms). Here, we introduce this subject and formalize the terminology about nontrivially recurrent points and nonwandering points for the context of planar piecewise smooth vector fields. A global bifurcation analysis of a special family of piecewise smooth vector fields presenting a nonwandering set with nontrivial recurrence is performed. As consequence, we are able to say that the Classical and the Improved Closing Lemmas are false for this scenario.

1. Introduction

One of the most challenging problems in the theory of dynamical systems (of continuous or discrete time) is the so called **Closing Lemma** (and variations thereof). Roughly speaking, in this problem the system has a nonperiodic point x_0 and the trajectory by x_0 return to a small neighborhood of x_0 infinitely many times. The objective is obtain a small perturbation of the original system in such a way that the new system has a closed trajectory through x_0.

This is an old problem (probably first stated by Poincaré in [16], vol 1, p. 82 in 1899) that have a lot of contributions along the history. This question is so relevant that Smale (in [22]) established it as one of the problems of the XXI century (in fact, it is the Problem 10). We strongly suggest reading the survey [1] about references and the main ideas behind this theme. Several versions of "Closing Lemmas" were stated with slightly distinct hypothesis and conclusions. As consequence several proofs were made (see [13, 14, 15, 17, 18, 19]) for these special cases and several cases were "Closing Lemmas" fail (see [10, 11, 20]) were stated.

A Piecewise Smooth Vector Field (PSVF, for short) $Z = (X, Y)$ on the plane is a pair of C^r-vector fields X and Y, where X and Y are restricted to regions of the plane separated by a smooth curve Σ (see the books [2] and [21] and references therein). In the context of PSVFs, a formal theoretical approach about minimal sets and chaos is in the beginning and only papers like [3, 4, 8, 12] address the issue. As far as the author knows, there are no

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papers dealing with any version of the Closing Lemma for PSVFs. So, here we introduce this investigation and establish some terminology. In special we stated Definitions 5 and 6 that formalize the concepts of nontrivially recurrent points, nonwandering points and distinguish two kinds of periodic points.

We also consider perturbations of a PSVF presenting a nonwandering set with nontrivial recurrence. All topological types in a neighborhood of such system are considered. The observation of such topological types, achieved by means of parametric piecewise smooth perturbations of the model, reveals that no one of them presents periodic points with the desired properties. As we shall explain below, in order to obtain such periodic points, even the traditional approach considering a local C^r-perturbation of some nonwandering/nontrivially recurrent point, known as a *C^r-surgery* (see Subsection 4.2.1) is unsuccessful.

The paper is organized as follow: In Section 2 we give a brief introduction to the PSVFs theory. In Section 3 we state the main results of the paper. In Section 4 we prove the main results. In Section 5 we give some conclusions about the paper.

2. Preliminaries

Now we formalize some basic concepts about PSVFs that pave the way in order to announce the main results. Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$ and consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. Observe that we can assume, locally around the origin of $\mathbb{R}^2$, that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases} \tag{1}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. Consider on Ω the product topology. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [9], that we summarize next. Indeed, consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .
(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s (see Figure 1). It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and then we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^Σ for both cases. In our pictures we represent the dynamics of Z^Σ by double arrows.

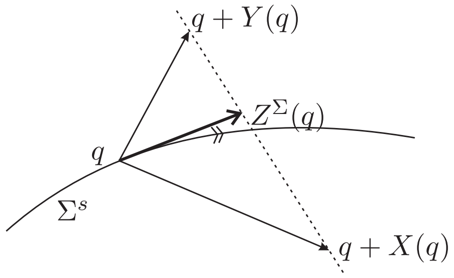

FIGURE 1. Filippov's convention.

We say that $q \in \Sigma$ is a Σ-*regular point* if

(i) $(X.f(q))(Y.f(q)) > 0$ or
(ii) $(X.f(q))(Y.f(q)) < 0$ and $Z^\Sigma(q) \neq 0$ (i.e., $q \in \Sigma^e \cup \Sigma^s$ and it is not an equilibrium point of Z^Σ).

The points of Σ which are not Σ-regular are called Σ-*singular*. We distinguish two subsets in the set of Σ-singular points: Σ^t and Σ^p. Any $q \in \Sigma^p$ is called a *pseudo-equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$ (q is a tangent contact point between the trajectories of X and/or Y with Σ).

Given a tangential singularity q, if there exist an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching q in a finite time, then such tangency is called a *visible tangency* for X (respec. Y); otherwise we call q an *invisible tangency* for X (respec. Y). In addition, a tangential singularity q is *singular* if q is a invisible tangency for both X and Y. On the other hand, a tangential singularity q is *regular* if it is not singular.

In what follows we present the definition of local and global trajectories for PSVFs. They were previously stated in [3] and [4].

Definition 1. *The* ***local trajectory (orbit)*** $\phi_Z(t,p)$ *of a PSVF given by* (1) *is defined as follows:*

- *For* $p \in \Sigma^+ \backslash \Sigma$ *and* $p \in \Sigma^- \backslash \Sigma$ *the trajectory is given by* $\phi_Z(t,p) = \phi_X(t,p)$ *and* $\phi_Z(t,p) = \phi_Y(t,p)$ *respectively.*

- *For $p \in \Sigma^c$ such that $X.f(p) > 0$, $Y.f(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_Y(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \geq 0$. For the case $X.f(p) < 0$ and $Y.f(p) < 0$ the definition is the same reversing time.*
- *For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \geq 0$. For $p \in \Sigma^s$ the definition is the same reversing time.*
- *For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \geq 0$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*
- *For p a singular tangency point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

Definition 2. *A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \Omega$ passing through p_0 is a union*

$$\Gamma_Z(t,p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$$

*of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory is a **positive** (respectively, **negative**) **global trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

Definition 3. *A set $A \subset \mathbb{R}^2$ is **positive-invariant** (respectively, **negative-invariant**) if for each $p \in A$ and all positive global trajectory $\Gamma_Z^+(t,p)$ (respectively, negative global trajectory $\Gamma_Z^-(t,p)$) passing through p it holds $\Gamma_Z^+(t,p) \subset A$ (respectively, $\Gamma_Z^-(t,p) \subset A$). A set $A \subset \mathbb{R}^2$ is **invariant** for Z if it is positive and negative-invariant.*

Definition 4. *Consider $\mathbf{Z} \in \mathbf{\Omega}$. A set $M \subset \mathbb{R}^2$ is **minimal** (respectively, either **positive-minimal** or **negative-minimal**) for Z if*

(i) *$M \neq \emptyset$;*
(ii) *M is compact;*
(iii) *M is invariant (respectively, either positive-invariant or negative-invariant) for Z;*
(iv) *M does not contain proper subset satisfying (i), (ii) and (iii).*

In [4] we proved that a positive and negative-minimal set is minimal, but the converse is false.

In the classical theory about vector fields, a **nontrivially recurrent point** is a point that is nonperiodic and belongs to the ω or α-limit set of itself and a **nonwandering point** is a point such that each one of its neighborhoods meets arbitrarily large iterations of itself (see [1]). Both these concepts can be adapted/extended to PSVFs. However, due to the non uniqueness of trajectories by points of $\Sigma^e \cap \Sigma^s$, it is possible the existence of amazing and unexploit behaviors of the trajectories of PSVFs. For example, in the classical theory about planar vector fields a trivial recurrent point is either an equilibrium point or belongs to a closed trajectory. For PSVFs this concept must be adapted since, as described in [3] and [4], there are PSVFs such that all points in

$$\Lambda = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \text{ and } x^4/2 - x^2/2 \leq y \leq 1 - x^2\}, \tag{2}$$

belong to a closed trajectory. So, following the classical definition, Λ is a trivially recurrent set, but the recurrence in Λ is far to be trivial. Moreover, given a point $p_0 \in \Lambda$ there are trajectories passing by p_0 with distinct periods or even trajectories that do not return to p_0. In fact, in [4] we shown that Λ is a nontrivial chaotic minimal set. Inspired on it, we give the following definitions:

Definition 5. *Let $Z \in \Omega$ and $p \in V$. Then*

(i) *p is a* ***single periodic point*** *for Z if there exists only one closed trajectory of Z by it,*
(ii) *p is a* ***multi periodic point*** *for Z if there exists more than one closed trajectories of Z by it.*

Remark 1. *Both kinds of points defined above are predicted in the literature. Single periodic points are pseudo equilibria or, according to the nomenclature in* [5, 6]*, belong to canard cycles. Multi periodic points are, for example, those ones of Λ (see* [3, 4]*).*

Definition 6. *Let $Z \in \Omega$ and $p \in V$. Then*

(i) *p is a* ***nontrivially recurrent point*** *for Z if it is not a single periodic point for Z and belongs to the ω or α-limit set of itself,*
(ii) *p is a* ***nonwandering point*** *for Z if each one of its neighborhoods meets arbitrarily large iterations of itself.*

Remark 2. *The concepts in Definition 6 coincide with those ones of smooth vector fields when Z is smooth. Both single periodic points and multi periodic points are nonwandering points and only multi periodic points are nontrivially recurrent points.*

3. Main Results

This work is part of a general program involving the asymptotic stability at typical singularities of systems represented by the following equation

$$\dot{u} = F(u) + sgn(u_n)G(u), \tag{3}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ and $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are smooth mappings.

Here we analyze the bifurcation diagram and the minimal sets of the following family of PSVFs:

(4)

$$Z_{\delta,\epsilon}(x,y) = \quad (\dot{x},\dot{y}) = \frac{1}{2}\Big((-1,-2(x-\delta)+x^2(4x+3)+(1+\epsilon)x(-3x-2)) \\ +sgn(y)\,(3,-2(x-\delta)-x^2(4x+3)-(1+\epsilon)x(-3x-2))\Big)$$

or, equivalently,

(5)

$$Z_{\delta,\epsilon}(x,y) = \begin{cases} X_\delta(x,y) = (1,-2(x-\delta)) & \text{if } y \geq 0, \\ Y_\epsilon(x,y) = (-2, x^2(4x+3)+(1+\epsilon)x(-3x-2)) & \text{if } y \leq 0, \end{cases}$$

with $\delta, \epsilon \in \mathbb{R}$ arbitrarily small parameters.

We introduce, for the planar PSVFs context, a very exploited subject in the classical theory about C^r-vector fields (and C^r-diffeomorphisms). We analyze nontrivially recurrent points and nonwandering points. Perturbations of PSVFs presenting such important objects are considered and we obtain the following results:

Theorem 7. *The bifurcation diagram of System* (5) *exhibits* 9 *distinct phase portraits (see the Bifurcation Diagram in Figure 11).*

By means of the analysis done in the proof of Theorem 7 we are able to obtain all topological types in a neighborhood of $Z_{0,0}$ given by (5) with $\delta = \epsilon = 0$. Since $Z_{0,0}$ has a nonwandering set with nontrivial recurrence, we can state, considering Definitions 5 and 6, that the following two Theorems are **FALSE** in the context of PSVFs, for all $r \geq 0$:

Theorem 8. ***(The Classical Closing Lemma for PSVFs)*** *Consider Z a PSVF presenting a nontrivially recurrent point p_0. Then, for every C^r-neighborhood $\mathcal{U}$ of Z, there exists a PSVF $\widetilde{Z}$ with single periodic point in p_0.*

Theorem 9. ***(The Improved Closing Lemma for PSVFs)*** *Consider Z a PSVF presenting a nonwandering nonperiodic point p_0. Then, for every C^r-neighborhood $\mathcal{U}$ of Z, there exists a PSVF $\widetilde{Z}$ with single periodic point in p_0.*

4. Proof of Main Results

4.1. **Bifurcations.** Our purpose in this section is to obtain a generic unfolding of the two-parameter family (5), exhibits its bifurcation diagram and characterize the minimal sets obtained in each choice of parameters. Note that the family (5) covers all topological types of PSVFs that can be obtained from the bifurcation of the minimal set Λ. Generically, the variation of the parameters λ and ϵ breaks the simultaneous tangential singularity of X_0 and Y_0 at the origin and/or the heteroclinic orbit passing through the visible tangency point of Y_0 (See Figure 2).

Let us distinguish some important points of the PSVF (5). The vector field X_δ has a unique tangential singularity situated at $p_1 = (\delta, 0)$. Moreover, p_1 is an invisible tangency. The vector field Y_ϵ has three tangential singularities situated at $q_1 = (\frac{1}{8}(3\epsilon - \sqrt{32 + 32\epsilon + 9\epsilon^2}), 0)$, $q_2 = (0, 0)$ and $q_3 = (\frac{1}{8}(3\epsilon - \sqrt{32 + 32\epsilon + 9\epsilon^2}), 0)$. Moreover, q_1 and q_3 are invisible tangencies and q_2 is a visible one.

By Lemma 4 of [7] the trajectories of X_δ are vertical translations of the graph of $h(x, \delta) = -(x - \delta)^2$, i.e., they are expressed by the curves

$$f(x, \delta, k_1) = -(x - \delta)^2 + k_1 \qquad (6)$$

with $k_1 \in \mathbb{R}$. Analogously, the trajectories of Y_ϵ are vertical translations of the graph of $g(x, \epsilon) = x^2(x + 1)(x - (1 + \epsilon))$, i.e., they are expressed by the curves

$$g(x, \epsilon, k_2) = x^2(x + 1)(x - (1 + \epsilon)) + k_2 \qquad (7)$$

with $k_2 \in \mathbb{R}$.

By (7) the Y_ϵ-trajectory by q_2 intersects transversally Σ at $q_0 = (-1, 0)$ and $q_4 = (1 + \epsilon, 0)$. By (6) the X_δ-trajectory by q_0 intersects (transversally) Σ again at $p_2 = (1 + 2\delta, 0)$ (See Figure 2).

Varying the parameters δ and ϵ we broke two unstable coincidences in Λ. The simultaneous occurrence of a tangency point of both X and Y (at the origin) is broken taking $\delta \neq 0$. Note that, if $\epsilon = 2\delta$ then $q_4 = p_2$ and the heteroclinic global trajectory by q_2 persists. So, this global closed trajectory is broken taking $\epsilon \neq 2\delta$. With this in mind, let us proceed the unfolding of the two-parameter family (5). In order to do it, consider the following notations:

γ_1	the Y_ϵ-orbit arc $\widehat{q_4\, q_0}$ connecting q_4 and q_0
γ_2	the X_δ-orbit arc $\widehat{q_0\, p_2}$ connecting q_0 and p_2
γ_3	the straight line $\overline{p_2\, q_4} \subset \Sigma$ connecting p_2 and q_4

• **Case 1: $\delta = 0$ and $\epsilon = 0$.** This case already was studied in [3] and [4]. Here there exists a nontrivial chaotic minimal set whose measure is not null. See Figure 2.

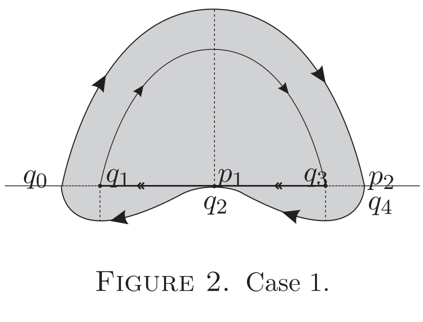

FIGURE 2. Case 1.

• **Case 2:** $\delta = 0$ **and** $\epsilon < 0$. In this case $p_1 = q_2$ and $q_4 < p_2$. So, the coincidence of fold points persists and the heteroclinic global trajectory by p_2 is broken. Let Λ_2 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_2 is negatively invariant and admits a proper subset Λ_2' negatively minimal. It is easy to see that Λ_2', the closure of the region bounded by $\widehat{q_4\, q_2} \cup \overline{q_2\, q_5} \cup \widehat{q_5\, q_6} \cup \widehat{q_6\, q_4}$, is a negatively minimal set, where $q_6 = \phi_X^-(q_4) \cap \Sigma$ and $q_5 = \phi_Y^-(q_6) \cap \Sigma$ with $\phi_W^-(x)$ being the negative W-trajectory by x. Note that Λ_2' is neither positively minimal nor minimal. Given, for example, $p_0 = (1+(\epsilon/2), -\epsilon^2/4)$ then there is not a closed trajectory by p_0. See Figure 3

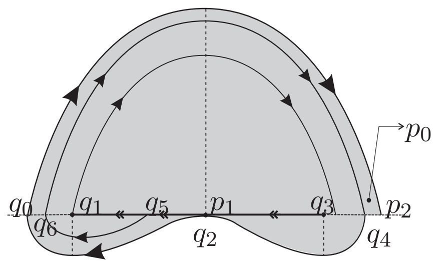

FIGURE 3. Case 2.

• **Case 3:** $\delta = 0$ **and** $\epsilon > 0$. In this case $p_1 = q_2$ and $q_4 > p_2$. So, the coincidence of fold points persists and the heteroclinic global trajectory by q_4 is broken. Let Λ_3 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_3 is positively invariant and admits a proper subset Λ_3' positively minimal. It is easy to see that Λ_3', the closure of the region bounded by $\widehat{q_2\, q_0} \cup \gamma_2 \cup \widehat{p_2\, q_7} \cup \overline{q_7\, q_2}$, is a positively minimal set, where $q_7 = \phi_Y^+(p_2) \cap \Sigma$ with $\phi_W^+(x)$ being the positive W-trajectory by x. Note that Λ_3' is neither negatively minimal nor minimal. Given, for example, $p_0 = (\epsilon, -\epsilon^2)$ then there is not a closed trajectory by p_0. See Figure 4.

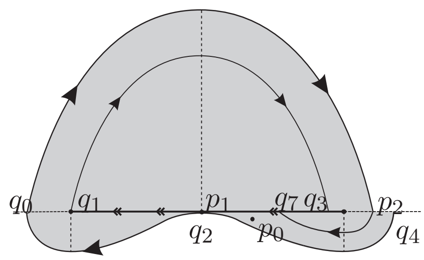

FIGURE 4. Case 3.

• **Case 4:** $\delta < 0$ **and** $\epsilon = \delta/2$. In this case $p_1 < q_2$ and $q_4 = p_2$. So, the coincidence of fold points is broken and the heteroclinic global trajectory by

$q_4 = p_2$ persists. Let Λ_4 be the closure of the region bounded by $\gamma_1 \cup \gamma_2$. According to Proposition 1 of [4], the set Λ_4 is minimal even if there exists a repeller pseudo equilibrium $p_3 = (a, 0)$ between q_1 and p_1, where

$$a = \delta/8 + ((96 - 48\delta - 9\delta^2)/(24\sqrt[3]{3}K_\delta)) - (K_\delta/(8\sqrt[3]{9})) \tag{8}$$

and K_δ is given by

$$\sqrt[3]{-9\delta(240 + \delta(8 + \delta)) + 8\sqrt{3}\sqrt{512 - 768\delta + 24540\delta^2 + 1700\delta^3 + 207\delta^4}}.$$

Given, for example, $p_0 = (0, -\delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_4)$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is $\partial\Lambda_4$ (respectively, p_3), with ∂B being the boundary of an arbitrary set B. See Figure 5.

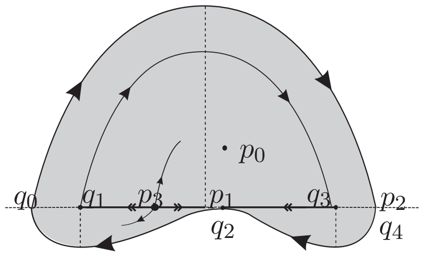

FIGURE 5. Case 4.

• **Case 5: $\delta > 0$ and $\epsilon = \delta/2$.** In this case $p_1 > q_2$ and $q_4 = p_2$. So, the coincidence of fold points is broken and the heteroclinic global trajectory by $q_4 = p_2$ persists. Let Λ_5 be the closure of the region bounded by $\gamma_1 \cup \gamma_2$. Analogously to the previous case, the set Λ_5 is minimal even if there exists an attractor pseudo equilibrium $p_3 = (a, 0)$ between p_1 and q_3, where a is given by (8). Given, for example, $p_0 = (0, \delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_5)$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is p_3 (respectively, $\partial\Lambda_5$). See Figure 6.

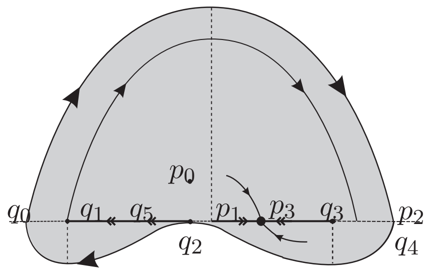

FIGURE 6. Case 5.

Cases 1−5 above represent structural unstable behaviors. The Cases 6−9 below represent structural stable behaviors.

• **Case 6:** $\delta < \mathbf{0}$ **and** $\epsilon < \delta/\mathbf{2}$**.** In this case $p_1 < q_2$ and $q_4 < p_2$. So, the coincidence of fold points and the heteroclinic global trajectory by q_2 are broken. Let Λ_6 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_6 is negatively invariant but it is not negatively minimal since there exists a negatively invariant subset $\{p_3 = (b, 0)\}$ between q_1 and p_1, where the expression of b is even more complicated than that one in (8) and we will omit it. The pseudo equilibrium p_3 is a repeller. Given, for example, $p_0 = (0, -\delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_6)$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is infinity (respectively, p_3). See Figure 7.

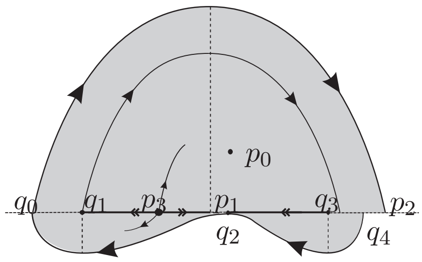

FIGURE 7. Case 6.

• **Case 7:** $\delta < \mathbf{0}$ **and** $\epsilon > \delta/\mathbf{2}$**.** In this case $p_1 < q_2$ and $q_4 > p_2$. So, the coincidence of fold points and the heteroclinic global trajectory by q_4 are broken. Let Λ_7 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_7 is positively invariant but it is not positively minimal since there exists a positively invariant subset Λ_7' defined as the closure of the region bounded by $\Upsilon_7 = \widehat{q_2\, q_0} \cup \gamma_2 \cup \widehat{p_2\, q_7} \cup \overline{q_7\, q_2}$. Note that Λ_7' is neither negatively minimal nor minimal. The cycle Υ_7 is attractor and has a repeller pseudo equilibrium $p_3 = (b, 0)$ at its interior. Given, for example, $p_0 = (0, -\delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_7')$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is Υ_7 (respectively, p_3). Moreover, in this case there are another amazing phenomenon characterized by *a stable connection between a pseudo equilibrium and a cycle.* See Figure 8.

• **Case 8:** $\delta > \mathbf{0}$ **and** $\epsilon < \delta/\mathbf{2}$**.** In this case $p_1 > q_2$ and $q_4 < p_2$. So, the coincidence of fold points and the heteroclinic global trajectory by p_2 are broken. Let Λ_8 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_8 is negatively invariant but it is not negatively minimal since there

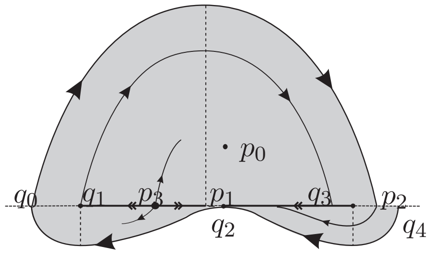

FIGURE 8. Case 7.

exists a negatively invariant subset Λ_8', defined as the closure of the region bounded by $\Upsilon_8 = \widehat{q_4\, q_2} \cup \overline{q_2\, q_5} \cup \widehat{q_5\, q_6} \cup \widehat{q_6\, q_4}$. The cycle Υ_8 is repeller and has an attractor pseudo equilibrium $p_3 = (b, 0)$ at its interior. Given, for example, $p_0 = (0, \delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_8')$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is p_3 (respectively, Υ_8). Again, we note a stable connection between a pseudo equilibrium and a cycle. See Figure 9.

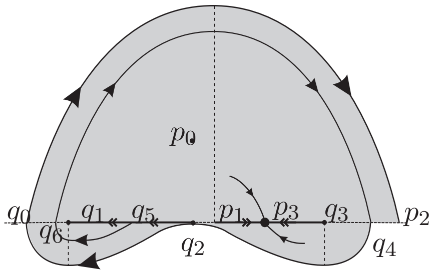

FIGURE 9. Case 8.

• **Case 9:** $\delta > 0$ **and** $\epsilon > \delta/2$. In this case $p_1 > q_2$ and $q_4 > p_2$. So, the coincidence of fold points and the heteroclinic global trajectory by q_4 are broken. Let Λ_9 be the closure of the region bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$. The set Λ_9 is positively invariant but it is not positively minimal since there exists a positively invariant subset $\{p_3 = (b, 0)\}$ between p_1 and q_3. The pseudo equilibrium p_3 is an attractor. Given, for example, $p_0 = (0, \delta)$ then there is not a closed trajectory by p_0. In fact, given any $p_0 \in Interior(\Lambda_9)$ then there is not a closed trajectory by p_0 because the ω-limit (respectively, α-limit) set of p_0 is p_3 (respectively, ∞). See Figure 10.

The bifurcation diagram of (5) is illustrated at Figure 11.

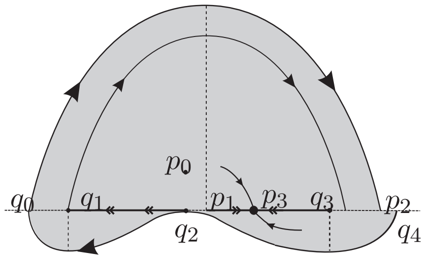

FIGURE 10. Case 9.

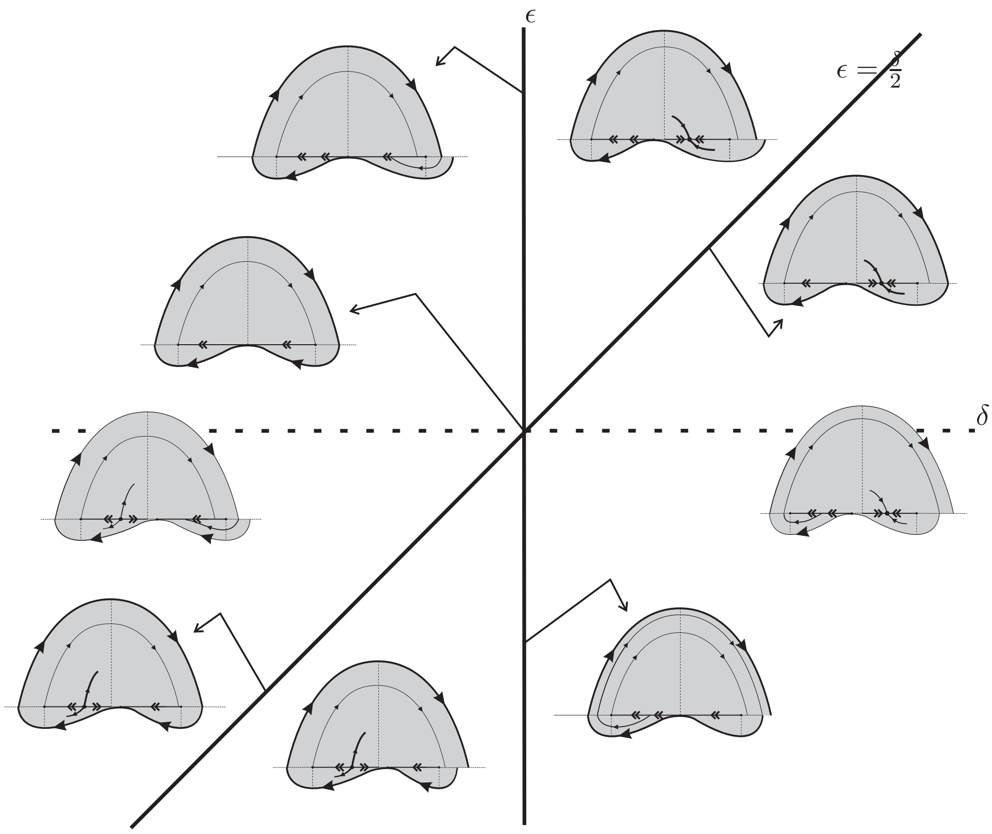

FIGURE 11. Bifurcation Diagram.

4.2. **Closing Lemma.** There are several versions of the Closing Lemma in the literature (for a non exhaustive list see [1]). The most popular of them are the following:

The Classical C^r Closing Lemma: *Consider X a C^r-vector field presenting a nontrivially recurrent point x_0. Then, for every C^r-neighborhood*

$\mathcal{U}$ of X in the set χ of all C^r-vector fields, there exists a C^r-vector field $\widetilde{X} \in \chi$ such that x_0 is a periodic point of $\widetilde{X}$.

The Improved C^r Closing Lemma: *Consider X a C^r-vector field presenting a nonwandering nonperiodic point x_0. Then, for every C^r-neighborhood $\mathcal{U}$ of X in the set χ of all C^r-vector fields, there exists a C^r-vector field $\widetilde{X} \in \chi$ such that x_0 is a periodic point of $\widetilde{X}$.*

In this section we study both the previous Closing Lemmas in the context of PSVFs and present two points of view for each one of them. In the first one we are looking for single periodic points and in the other one we are searching multi periodic points.

Let us prove now that an analogous to the Classical (respectively, Improved) Closing Lemma is false when we are searching for single periodic points. In fact, consider $Z = Z_{0,\epsilon}$ given by (5) with $\delta = 0$. Take $p_0 = (x_0, y_0) \in \Sigma^+$, with y_0 sufficiently small. Since there exists a periodic orbit by p_0, in order to have p_0 a nonperiodic point of some PSVF W we consider $W = Z|_{V/\{w_0\}}$, where $w_0 = (-\sqrt{x_0^2 + (y_0/2)}, y_0/2)$. It is easy to see that w_0 belongs to $\phi_X^-(p_0)$, w_0 is placed between Σ^e and p_0 and p_0 is a nontrivially recurrent (respectively, nonwandering nonperiodic) point of $Z|_{V/\{w_0\}}$. See Figure 12. Moreover, since we are excluding w_0, there is not a closed trajectory of $Z|_{V/\{w_0\}}$ through p_0.

Consider now a small perturbation of $Z|_{V/\{w_0\}}$. According to Section 4.1, all perturbations breaking the simultaneous tangency point at the origin do not produce single periodic orbits. So, the last expectancy is that a single periodic point appears in a perturbation that keeps the simultaneous tangency point. In this context, let $s_0 \in \widehat{w_0\, p_0}$, where $\widehat{w_0\, p_0}$ is the arc of trajectory connecting w_0 and p_0. So, a classical C^r-surgery (or, according to [1], pg 1657, a *motion of a point into s_0 in the δ-core of a ball*) in a small neighborhood of s_0 can produces a closed trajectory Γ_0 through p_0. Let us construct such surgery.

4.2.1. *Construction of the C^r-surgery:* Consider $\mathcal{W}$ a small neighborhood of $s_1 \in \widehat{w_0\, s_0}$ and $s_2 \notin \phi_X^-(s_0)$ such that $s_2 \in \mathcal{W}$. Suppose that we are able to connect s_2 and s_0 by an arc of trajectory γ_4 in such a way that the new trajectory through p_0 is a C^r-curve defined in Σ^+ (if such a construction is not possible then it is straightforward that the Theorems 8 and 9 are false). See Figure 12. Take $l_1 = \phi_X^-(s_2) \cap \Sigma^e$ and $l_2 \neq l_1$ placed at the straight line segment $\overline{l_1\, q_2}$. So, while $\Gamma_0 = \widehat{p_0\, l_1} \cup \widehat{l_1\, s_2} \cup \gamma_4 \cup \widehat{s_0\, p_0}$ has period T_0, it is easy to construct a closed trajectory $\Gamma_1 = \widehat{p_0\, l_2} \cup \widehat{l_2\, q_2} \cup \widehat{q_2\, l_1} \cup \widehat{l_1\, s_2} \cup \gamma_4 \cup \widehat{s_0\, p_0}$ with period $T_1 > T_0$ (see Figure 12). So, p_0 is not a single periodic point. This proves that Theorem 8 (respectively, Theorem 9) is false.

4.2.2. *Forthcoming papers:* Consider the following conjectures:

Conjecture 1. ***(The Classical Closing Lemma for multi periodic points of PSVFs)*** *Consider Z a PSVF presenting a nontrivially recurrent point p_0. Then, for every C^r-neighborhood $\mathcal{U}$ of Z, there exists a PSVF $\widetilde{Z}$ with multi periodic point in p_0.*

Conjecture 2. ***(The Improved Closing Lemma for multi periodic points of PSVFs)*** *Consider Z a PSVF presenting a nonwandering nonperiodic point p_0. Then, for every C^r-neighborhood $\mathcal{U}$ of Z, there exists a PSVF $\widetilde{Z}$ with multi periodic point in p_0.*

Note that the previous argument does not produces a contradiction with Conjectures 1 and 2. So, these conjectures can be true. However, by now, we are not able to construct explicitly neither the C^r-surgery producing the closed trajectory Γ_0 nor a general argument that works well for every nontrivial recurrent or nonwandering nonperiodic point. We hope to prove these results in a forthcoming paper.

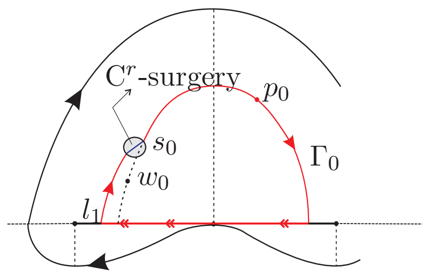

FIGURE 12. Single periodic point p_0.

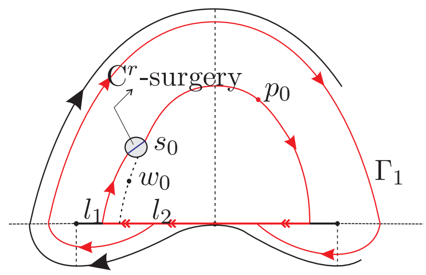

FIGURE 13. Multi periodic point p_0.

5. Conclusion

In this paper we are concerned with a global analysis of PSVFs in $\mathbb{R}^2$ in the sense of obtaining periodic points in PSVFs that are small perturbations of a PSVF presenting nontrivially recurrent points or nonwandering points. We obtain a PSVF $Z_{0,0}$, given in (5) with $\delta = \epsilon = 0$, presenting a nonwandering set Λ with nontrivial recurrence (in [3] and [4] we prove that Λ is a non trivial chaotic minimal set of $Z_{0,0}$). We exhibit all topological types around $Z_{0,0}$. Between these topological types we highlight a stable connection between a pseudo equilibrium and a cycle (Cases 7 and 8 of Subsection 4.1) and cases where a nonwandering/nontrivial non null measure set persists after a perturbation (Cases 2 and 3 of Subsection 4.1). Also we conclude that an analogous to the Classical and the Improved Closing Lemmas can not be obtained for PSVFs in the sense of finding single periodic points, but they can be possible in the sense of finding multi periodic points. We intent perform this last issue in a forthcoming paper.

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[1] FC–UNESP, CEP 17033–360, Bauru, São Paulo, Brazil
E-mail address: tcarvalho@fc.unesp.br