EQUIVALENCE BETWEEN POSITIVE LEBESGUE MEASURE MINIMAL AND CHAOTIC SETS FOR PLANAR PIECEWISE SMOOTH VECTOR FIELDS

TIAGO DE CARVALHO[1] AND RODRIGO DONIZETE EUZÉBIO[2]

ABSTRACT. Despite of the fact that piecewise smooth vector fields have been used to model a large range of physical phenomena and other applications, there exists nowadays an evident lack in a formal approach about minimal and chaotic sets inherent to these kind of dynamical systems. Here we state some new terminology and results that give a final conclusion about the connection between both concepts on bi-dimensional spaces. Moreover, since there is not uniqueness of trajectories for piecewise smooth vector fields, we show that the orientation induced on the trajectories plays an essential role at the qualitative analysis of these objects.

1. INTRODUCTION

Since the outstanding work of Poincaré (see [23]) until nowadays the theory of dynamical systems have become increasingly important in physics and mathematics and today it involves not only geometrical and experimental approaches but also topological, analytical and algebraic ones. Indeed, a today well accepted dissolution of dynamical systems theory highlights discrete, continuous and ergodic aspects of the maps and sets involved in the study of a given problem. Of course, there are intersections between each approach although a general theory taking into account general aspects seems to be far away to be reached. In particular, inside the continuous approach has emerged a new theory which takes into account not smooth but *piecewise smooth vector fields* (PSVFs, for short), that is, vector fields which are not smooth everywhere but non-smooth or even discontinuous in some regions of the geometrical space. Such special systems have found several applications in applied sciences (see the books [1] and [24] and references therein). In fact, as stated in [9], piecewise smooth phenomena occur in physical systems that operate according to distinct kinds of behavior. The transition from one kind of behavior to another one can be idealized as a discrete and instantaneous transition. For example, a non exhaustive list of applications of such theory involves mechanical systems (see [4], [12], [19]), electrical circuits (see [2, 18]), the stick-slip process (see [15]), relay systems (see [3] and [16]) and control theory (see [11]). Since the transition of one behavior to another one is faster than the dynamic of the system, the transition can be modeled as instantaneous. For example, the physical characteristics of a diode or the mathematical model of dry friction in mechanical systems, switching between the sliding mode (slip) and the grip (stick).

2010 *Mathematics Subject Classification.* Primary 34C28, 37D45.
Key words and phrases. vector fields, piecewise smooth vector fields, chaos, minimal sets.

PSVFs have attracted the attention from the mathematical community just a few decades ago, having as one of the landmarks the works of Teixeira concerning manifolds with boundary (see [25]). Later, Filippov (see [14]) stated several aspects regarding the methodology of PSVFs. In summary, PSVFs are objects defined not by a unique system but two or more, each one defined in a region of the space which is separated by a codimension one manifold currently called *switching* or *discontinuity manifold*. Moreover, it is supposed that under this manifold we have defined one or more vector fields (the adjacent ones), so apart from special cases the trajectory of a PSVF is not unique. Indeed, it is basically the fundamental feature of PSVFs which leads to a more complex and abstract theory than that one developed for smooth or continuous dynamical systems.

A question that arise concerning planar PSVFs regards its discrete or ergodic aspects and, surely, the validity of results coming from the continuous context into this particular scenario. In the last case, we should mention, for instance, the work of Sotomayor and Machado (see [20]) which provide the conditions in order to assure the validity of the acclaimed Peixoto Theorem for planar PSVFs (see [22]). We also mention [5] where the authors provide a version of the classical Poincaré-Bendixson Theorem in the planar PSVF scenario. On the other hand, regarding ergodic properties of planar PSVF, in [5] the authors also provide an example of a bi-dimensional set which is a positive measure non-trivial minimal set for a planar PSVF, which means that such objects may occur by suppressing the smoothness of the system. Indeed, as far as the authors know, it was one of the first works connecting concepts coming from the ergodic theory of dynamical systems to PSVFs.

Here we observe the importance of take into account not only the switching manifold but also the orientation of the trajectories on the time. Indeed, the present work is the continuation of these two previous works [5, 6] and goes forward in the sense of establish the concepts of the ergodic theory of dynamical systems for PSVFs. Other results following the same aim can be found in [8] and [13], as well as the the works [10] and [17].

More specifically, in this paper we introduce the concept of orientable chaos and observe the presence of some sets which are orientable chaotic in the sense of Definition 9 but not properly chaotic. It stress the necessity of taking into account the orientation of the objects, as commented before. Moreover, we provided a sufficient condition ir order to a non null measure compact invariant set be minimal in terms of it chaoticity, which is the converse of the Theorem 14 of [6]. With this, we establish a final connection between minimal sets and chaotic PSVFs. So, from the results of this paper one should note that important phenomena occur by taking into account sets having positive Lebesgue measure when studying ergodic aspects of PSVFs, which find an important place for some areas of physics as statistical physics, apart from ergodic theory.

Our main results are the following:

Theorem 1 *A PSVF Z is chaotic on W if, and only if, Z is positive chaotic and negative chaotic on W.*

The next theorem says, among other things, that Theorem 1 only makes sense if $med(W) > 0$ where $med(\cdot)$ is the Lebesgue measure. We have the following result:

Theorem 2 *Let $K \subset \mathbb{R}^2$ be a compact invariant set and Z a PSVF. If $med(K) = 0$ then Z is not chaotic on K.*

Theorem 3 *If Z is chaotic on W and $med(W) > 0$ then W is positive minimal and negative minimal.*

Other important result, which actually is a corollary of Theorem 3 proves that if $med(W) > 0$ and Z is chaotic on W then W is minimal for Z (see Corollary 1).

Summarizing, let W be a non null measure set and Z a PSVF defined in W. Then, according to Theorem 14 of [6] and Theorems 1 and 3 we get:

$$\begin{array}{c} Z \text{ is pos. and} \\ \text{neg. chaotic on } W \end{array} \Leftrightarrow \begin{array}{c} Z \text{ is chaotic} \\ \text{on } W \end{array} \Leftrightarrow \begin{array}{c} W \text{ is pos. and} \\ \text{neg. min. for } Z \end{array} \Rightarrow \begin{array}{c} W \text{ is min.} \\ \text{for } Z \end{array}$$

Some particular conclusions we get from the results are the following. First, we note that although the chaoticity of a PSVF Z under a set W implies that W is minimal for Z, the converse is false according to Example 2 of [6]. Also, if Z is positive (resp. negative) chaotic on W then W is positive (resp. negative) minimal for Z (see Corollary 3), but the converse is false since we can exhibit positive (resp. negative) minimal sets that are not positive (resp. negative) chaotic (see Example 1). Other considerations and results are presented timely throughout the text.

The paper is organized as follow: In Section 2 we give some standard definitions about PSVFs theory. In Section 3 we setting the problem, give specific definitions for our context and illustrate the theory with some examples. In Section 4 we prove the main results and consequences of them.

2. Preliminaries

Now we formalize some definitions and conventions about PSVFs that pave the way in order to prove the main results. Those results are kind standard and can be find in papers like [5, 6, 8] among others.

Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$ and consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). The switching manifold Σ is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. Observe that we can assume, locally around the origin of $\mathbb{R}^2$, that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$Z(x,y) = \left\{ \begin{array}{ll} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{array} \right. \tag{1}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. Consider on Ω the product topology. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [14], that we summarize next. Indeed, consider the Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .
(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.
(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s (see Figure 1). It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and then we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^Σ for both cases. In our pictures we represent the dynamics of Z^Σ by double arrows.

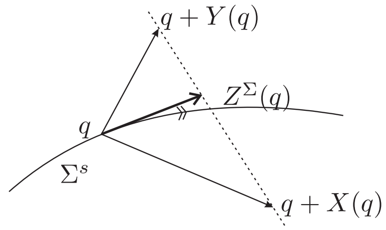

FIGURE 1. Filippov's convention.

We say that $q \in \Sigma$ is a Σ-*regular point* if
(i) $(X.f(q))(Y.f(q)) > 0$ or
(ii) $(X.f(q))(Y.f(q)) < 0$ and $Z^\Sigma(q) \neq 0$ (i.e., $q \in \Sigma^e \cup \Sigma^s$ and it is not an equilibrium point of Z^Σ).

The points of Σ which are not Σ-regular are called Σ-*singular*. We distinguish two subsets in the set of Σ-singular points: Σ^t and Σ^p. Any $q \in \Sigma^p$ is called a *pseudo-equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$ (q is a tangent contact point between the trajectories of X and/or Y with Σ).

Given a tangential singularity q, if there exist an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching q in a finite time, then such tangency is called a *visible tangency* for X (respec. Y); otherwise we call q an *invisible tangency* for X (respec. Y). In addition, a tangential singularity q is *singular* if q is a invisible tangency for both X and Y. On the other hand, a tangential singularity q is *regular* if it is not singular.

In what follows we present the definition of local and global trajectories for PSVFs. They were previously stated in [5] and [6].

Definition 1. *The* ***local trajectory (orbit)*** $\phi_Z(t,p)$ *of a PSVF given by* (1) *is defined as follows:*

- *For* $p \in \Sigma^+ \backslash \Sigma$ *and* $p \in \Sigma^- \backslash \Sigma$ *the trajectory is given by* $\phi_Z(t,p) = \phi_X(t,p)$ *and* $\phi_Z(t,p) = \phi_Y(t,p)$ *respectively.*
- *For* $p \in \Sigma^c$ *such that* $X.f(p) > 0$, $Y.f(p) > 0$ *and taking the origin of time at* p, *the trajectory is defined as* $\phi_Z(t,p) = \phi_Y(t,p)$ *for* $t \leq 0$ *and* $\phi_Z(t,p) = \phi_X(t,p)$ *for* $t \geq 0$. *For the case* $X.f(p) < 0$ *and* $Y.f(p) < 0$ *the definition is the same reversing time.*

- *For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \geq 0$. For $p \in \Sigma^s$ the definition is the same reversing time.*
- *For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \geq 0$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*
- *For p a singular tangency point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

Definition 2. *A* ***global trajectory (orbit)*** *$\Gamma_Z(t,p_0)$ of $Z \in \Omega$ passing through p_0 is a union*

$$\Gamma_Z(t,p_0) = \bigcup_{i\in\mathbb{Z}} \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$$

of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory Γ_Z^+ is a ***positive*** *(respectively, Γ_Z^-* ***negative****)* ***global trajectory*** *if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

Definition 3. *A set $A \subset \mathbb{R}^2$ is* ***positive invariant*** *(respectively,* ***negative invariant****) if for each $p \in A$ and all positive global trajectory $\Gamma_Z^+(t,p)$ (respectively, negative global trajectory $\Gamma_Z^-(t,p)$) passing through p it holds $\Gamma_Z^+(t,p) \subset A$ (respectively, $\Gamma_Z^-(t,p) \subset A$). A set $A \subset \mathbb{R}^2$ is* ***invariant*** *for Z if it is positive and negative invariant.*

Definition 4. *Consider $\mathbf{Z} \in \mathbf{\Omega}$. A set $M \subset \mathbb{R}^2$ is* ***minimal*** *(respectively, either* ***positive minimal*** *or* ***negative minimal****) for Z if*

(i) *$M \neq \emptyset$;*
(ii) *M is compact;*
(iii) *M is invariant (respectively, either positive invariant or negative invariant) for Z;*
(iv) *M does not contain proper subset satisfying (i), (ii) and (iii).*

3. Orientable chaotic PSVFs

3.1. **Setting the problem.** One of the most important facts concerning PSVFs is the orientation of its trajectories. Indeed, it is very important, for instance, for the concept of invariance or for define the flow associated to the Filippov vector field. Of course, in the smooth theory of vector fields this distinction does not play an important role since we have uniqueness of trajectories.

In this direction, we should verify if such distinction is also necessary when defining chaotic PSVFs. Indeed, as we will see later in this section, chaotic systems does not play the same role by considering positive and negative times. Nevertheless, in this section we present examples of chaotic systems either for positive or negative trajectories which are not chaotic systems. As far as the authors know, this is the first time that such approach is presented in the literature about PSVFs, although the concept of chaos in such systems have been discussed before, for instance, in [6] and [10].

Of course, the definition of chaos contemplates topological transitivity and sensitivity to initial conditions. For this reason, we must also introduce the definition of transitivity and sensible dependence in their *orientable* versions. Next definitions take into account the previous discussion.

3.2. **Definitions concerning chaotic aspects.**

Definition 5. *System* (1) *is* ***topologically transitive*** *on an invariant set W if for every pair of nonempty, open sets U and V in W, there exist $q^+, q^- \in U$, $\Gamma_Z^+(., q^+), \Gamma_Z^-(., q^-)$ global trajectories and $t_0^+ > 0 > t_0^-$ such that $\Gamma_Z^+(t_0^+, q^+)$ and $\Gamma_Z^-(t_0^-, q^-) \in V$.*

Considering either the positive or the negative orientation for the trajectories of Z, we get:

Definition 6. *System* (1) *is* ***topologically positive transitive*** *(respectively,* ***topologically negative transitive****) on a positive invariant (respectively, negative invariant) set W if for every pair of nonempty, open sets U and V in W, there exist $q \in U$, $\Gamma_Z^+(t, q)$ a positive (respectively, $\Gamma_Z^-(t, q)$ a negative) global trajectory and $t_0 > 0$ (resp., $t_0 < 0$) such that $\Gamma_Z^+(t_0, q) \in V$ (resp., $\Gamma_Z^-(t_0, q) \in V$).*

Analogously to the definition of topologically transitive systems, the definition of sensitive dependence for PSVFs should be as follows. Note that there is an important difference concerning the invariance of the sets: they must be positive or negative invariant when defining concepts taking into account their orientation.

Definition 7. *System* (1) *exhibits* ***sensitive dependence*** *on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < diam(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist $y^+, y^- \in B_\varepsilon(x) \cap W$ and global trajectories Γ_x^+, Γ_x^-, $\Gamma_{y^+}^+$ and $\Gamma_{y^-}^-$ passing through x, y^+ and y^-, respectively, satisfying*

$$d_H(\Gamma_x^+, \Gamma_{y^+}^+) = \sup_{a \in \Gamma_x^+, b \in \Gamma_{y^+}^+} d(a, b) > r,$$

and

$$d_H(\Gamma_x^-, \Gamma_{y^-}^-) = \sup_{a \in \Gamma_x^-, b \in \Gamma_{y^-}^-} d(a, b) > r,$$

where $diam(W)$ is the diameter of W and d is the Euclidean distance.

Associated to the previous definition we give the next one, where the orientation of the trajectories of Z also is considered:

Definition 8. *System* (1) *exhibits* ***sensitive positive dependence*** *(resp.,* ***sensitive negative dependence****) on a compact positive invariant (resp., negative invariant) set W if there is a fixed $r > 0$ satisfying $r < diam(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive (resp., negative) global trajectories Γ_x^+ and Γ_y^+ (resp., Γ_x^- and Γ_y^-) passing through x and y, respectively, satisfying*

$$d_H(\Gamma_x^+, \Gamma_y^+) = \sup_{a \in \Gamma_x^+, b \in \Gamma_y^+} d(a, b) > r$$

$$\left(\text{ resp., } d_H(\Gamma_x^-, \Gamma_y^-) = \sup_{a \in \Gamma_x^-, b \in \Gamma_y^-} d(a, b) > r \right),$$

where $diam(W)$ is the diameter of W and d is the Euclidean distance.

In this paper we will consider the notations stated in the following table.

Table of Abbreviations	
Topologically transitive	TT
Topologically positive transitive	TPT
Topologically negative transitive	TNT
Sensitive dependence	SD
Sensitive positive dependence	SPD
Sensitive negative dependence	SND

We should mention, as observed in [10], that Definitions 5 and 7 coincide with the definitions of topological transitivity and sensible dependence of smooth vector fields for single-valued flows, so these definitions are natural extension for a set-valued flow.

Following we introduce the definition of a chaos and orientable chaos in the piecewise smooth context:

Definition 9. *System* (1) *is* ***chaotic*** *(resp., either* ***positive chaotic*** *or* ***negative chaotic****) on a compact invariant (resp., either positive invariant or negative invariant) set W if it is TT and exhibits SD (resp., either TPT and exhibits SPD or TNT and exhibits SND) on W.*

3.3. **Examples of orientable chaotic sets.**

3.3.1. **Orientable chaotic sets that are not chaotic:** Consider the PSVF:

$$\begin{aligned} Z_\epsilon(x,y) = \quad & (\dot{x},\dot{y}) = \tfrac{1}{2}\Big((-1,-2x+x^2(4x+3)+(1+\epsilon)x(-3x-2)) \\ & \qquad +sgn(y)\,(3,-2x-x^2(4x+3)-(1+\epsilon)x(-3x-2))\Big) \end{aligned} \tag{2}$$

or, equivalently,

$$Z_\epsilon(x,y) = \left\{ \begin{array}{ll} X(x,y) = (1,-2x) & \text{if } y \geq 0, \\ Y_\epsilon(x,y) = (-2, x^2(4x+3)+(1+\epsilon)x(-3x-2)) & \text{if } y \leq 0, \end{array} \right. \tag{3}$$

with $\epsilon \in \mathbb{R}$ an arbitrarily small parameter. In [6] the authors proved that Z_0 has a chaotic set given (see Figure 2) by

$$\Lambda = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \;\text{ and }\; x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \tag{4}$$

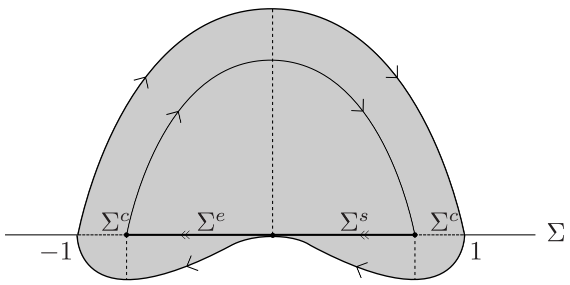

FIGURE 2. Chaotic set Λ.

Taking $\epsilon < 0$ (resp., $\epsilon > 0$) in (3) the PSVF Z_ϵ has a negative chaotic (resp., positive chaotic) set $\widetilde{\Lambda}$, see the shadowed region in Figure 3 (resp., Figure 4), bounded by $\widehat{p_1\,p_2} \cup \widehat{p_2\,p_3} \cup \widehat{p_3\,p_4} \cup \widehat{p_4\,p_1}$, where $\widehat{a\,b}$ is the orbit-arc connecting the points a and b. Despite of this, when $\epsilon \neq 0$, $\widetilde{\Lambda}$ is not a chaotic set. This happens because $\widetilde{\Lambda}$ is not an invariant set; it is only negative invariant (resp., positive invariant).

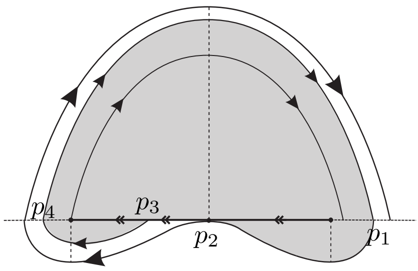

FIGURE 3. Negative chaotic set $\widetilde{\Lambda}$.

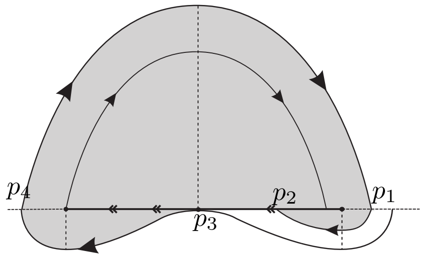

FIGURE 4. Positive chaotic set $\widetilde{\Lambda}$.

Remark 1. *The previous paragraph remains true if we change the word chaotic by the word minimal. A complete bifurcation analysis of the family* (3) *is given in* [8].

3.3.2. **Orientable chaotic sets and orientable minimality:** The sets given in Figures 3 and 4 are orientable chaotic and orientable minimal sets. Despite of this, it is easy to exhibit examples of orientable minimal sets that are not orientable chaotic.

Example 1. *Consider the PSVF*

$$Z(x,y) = (X(x,y), Y(x,y)) = ((-1, 3x^2-3), (1, -(9/4)+3(-1+x)x)).$$

Such PSVF has a periodic orbit (see Figure 5) wich is a negative minimal set. However, Z is not a negative chaotic PSVF on the periodic orbit since it does not present SPD.

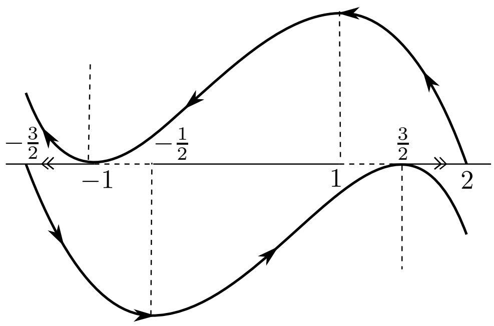

FIGURE 5. Periodic orbit (for positive time)

Observe that, in the last example the Lebesgue measure of the periodic orbit is null. However, it is not difficult to exhibit a minimal sets W for some PSVF, with $med(W) > 0$, in such way that W is neither positive chaotic nor negative chaotic. Indeed, Example 2 of [6] satisfies these properties. In other words, in general minimality does not imply chaoticity. The converse, on the other hand, is true, as proved in Section 4.

4. Main Results

In this Section we present the main results of the paper.

Proposition 1. *Let Z be a PSVF. The following statements hold*

(a) *Z is TT on W if, and only if, Z is simultaneously TPT and TNT on W;*
(b) *Z exhibits SD on W if, and only if, Z exhibits simultaneously SPD and SND on W;*

An analogous of Proposition 1 does not hold for minimal sets. Indeed, while sets which are both positive and negative minimal are also minimal, the converse is not true. Again, Example 2 of [6] exemplify this situation.

Following we prove Proposition 1.

Proof of Proposition 1. We start proving item (a). First, assume that Z is TT on W and consider disjoint open sets U and V contained in W. Consequently, there exist points p_+ and p_- in U, times $t_+ > 0$ and $t_- > 0$ and trajectories Γ^+ and Γ^- satisfying $\Gamma^+(t_+, p_+) \in V$ and $\Gamma^-(-t_-, p_-) \in V$, where Γ^+ and Γ^- are positive and negative trajectories passing, respectively, through the points p_+ and p_-. Moreover, according to Remark 1 of [6], W is simultaneously positive and negative invariant. Consequently U and V can be connected through the positive trajectory Γ^+ starting on p_+ after a time t_+, that is, Z is TPT on W. Analogously Z is TNT on W.

The converse is straightforward according to Definitions 5 e 6.

The proof of the item (b) follow exactly the same ideas exposed at the proof of item (a). □

Proposition 1 allows us to prove the following result.

Theorem 1. *A PSVF Z is chaotic on W if, and only if, Z is positive chaotic and negative chaotic on W.*

Proof of Theorem 1. First, assume that Z is chaotic on W. By Definition 9, Z is TT on W and exhibits SD on W. Thus, by Proposition 1, Z is simultaneously TPT and TNT on W and exhibits both SPD and SND on W. Then, using again Definition 9, we obtain that Z is positive chaotic and negative chaotic on W. The converse is straightforward. □

The most part of the results obtained in [5] and [6] takes into account sets having positive Lebesgue measure. Indeed, in almost every approach concerning ergodic aspects of PSVFs, this is the interesting case. We cite, for instance, the existence of non-trivial minimal sets and planar chaotic PSVFs, as shown in the papers cited previously. Indeed, as states the next theorem, there is no sense in considering chaotic systems with null Lebesgue measure.

Theorem 2. *Let $K \subset \mathbb{R}^2$ be a compact invariant set and Z a PSVF. If $med(K) = 0$ then Z is not chaotic on K.*

Proof. First, suppose that $K \cap \Sigma \subset \Sigma^c$ and take $p \in K$. Consequently, the flow associated to Z, namely $\phi_t^Z(p)$ with $\phi_0^Z(p) = p$, satisfies $\phi_t^Z(p) \xrightarrow{t\to\infty} \Omega \in \omega(p) \subset K$, since K is compact. Here $\omega(p)$ denotes the ω-limit set of the point p. Thus, by using Poincaré Bendixson Theorem for PSVFs (see [5]) we get that Ω is a (pseudo-)equilibrium, a (pseudo-)cycle or a (pseudo-)graph. In any case, it is trivial to see that Z is not chaotic on K since Z does not exhibits SD on K.

Now consider the case where $K \cap (\Sigma^s \cup \Sigma^e) \neq \emptyset$ and suppose that there exist a PSVF Z which is chaotic on K. Take $p \in K \cap (\Sigma^s \cup \Sigma^e)$ and $V_p \subset \mathbb{R}^2$ a neighborhood of p. We will show that K is not invariant, more specifically, that there exist a trajectory $\tilde{\phi}_t$ passing through p and $t^* \in \mathbb{R}$ such that $\phi_{t^*}(\tilde{p}) = p$ with $\tilde{p} \notin K$. Indeed, consider the sets $V_p^+ = \{\phi_t^+(p) \cap V \mid \phi_t^+$ is the positive trajectory of Z passing through $q\}$ and V_p^- defined analogously for the negative trajectory. Observe that $med(V_p^+ \cup V_p^-) > 0$, since using the Definition 1, in this case the preimage of $K \cap (\Sigma^s \cup \Sigma^e)$ contain an open set $U \subset V_p$ satisfying $0 < med(U) < med(V_p^+ \cup V_p^-)$. Consequently there exist a point $q \in V_p^+ \cup V_p^-$ such that $q \notin K$, because otherwise $V_p^+ \cup V_p^- \subset K$ and then $med(K) > med(V_p^+ \cup V_p^-) > 0$ (see Figure 6).

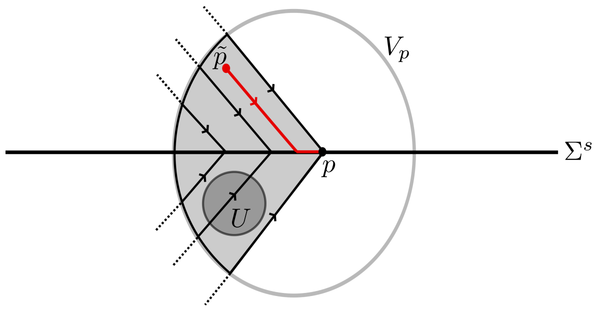

FIGURE 6. The neighborhood V_p of p. The filled region correspond to V_p^-, and in this case $V_p^+ = V_p \cap \Sigma$. Observe that it has positive Lebesgue measure. The trajectory in red correspond to $\tilde{\phi}_t$.

□

We observe that Theorem 2 does not make sense if Z is smooth, once there is no bi-dimensional smooth chaotic flow.

Before announce Theorem 3 let us prove the following lemma thet will be useful in its proof, besides being elegant itself.

Lemma 1. *Let Z a chaotic PSVF on W. Then Z is chaotic on every compact invariant proper subset $\widetilde{W} \subset W$.*

Proof. Suppose that Z is not chaotic on $\widetilde{W}$. So, by Definition 9 we get that Z is not TT or does not presents SD on $\widetilde{W}$. In any case, since $\widetilde{W} \subset W$ we also get that Z is not TT or does not presents SD on W. This is a contradiction with the hypothesis. □

In [6], among other results, the authors prove that, if a compact invariant set W satisfying $med(W) > 0$ is simultaneously positive and negative minimal for a PSVF Z, then Z is chaotic on W. Now, we prove the converse of this important theorem, as says Theorem 3 in what follows. Observe that, due to Theorem 2, we must impose a condition demanding the positive Lebesgue measure of the considered set.

Theorem 3. *If Z is chaotic on the compact invariant set W and $med(W) > 0$, then W is positive minimal and negative minimal for Z.*

Proof. According to Theorem 1, Z is positive chaotic on W. So, W is compact, non-empty and positive invariant. Suppose that W is not positive minimal. In this case, there exists a proper subset $\widetilde{W}$ of W with the previous three properties. Moreover, by Lemma 1 and Theorem 2, we get $med(\widetilde{W}) > 0$. Of course $\widetilde{W}$ is not dense in W since $\widetilde{W}$ is compact and $\widetilde{W} \neq W$. Therefore there exists an open set $A \subset W$ such that $A \cap \widetilde{W} = \emptyset$. We can take A in such a way that $med(A) < med(W)/2$. Let $B \subset \widetilde{W}$ an open set of W (this is possible because $med(A) < med(W)/2$). In this way, using the open sets A and B, we have that Z is not TPT. But this is a contradiction with the fact that Z is chaotic on W. Therefore, W is positive minimal for Z.

An analogous argument proves that W is negative minimal for Z. □

Next corollary is a straightforward consequence of Theorem 3, but it is very important once it provides a ultimate answer about the relation between chaotic systems and minimal sets.

Corollary 1. *If Z is chaotic on W and $med(W) > 0$ then W is minimal for Z.*

Proof. It is enough to use Theorem 3 and Lemma 2 of [6]. □

We remark that the converse is not true, as observed in [6].

Next two corollaries are also consequences of Theorem 3. Their proof, analogously, are quite trivial although the results can find applications.

Corollary 2. *If $med(W) > 0$ and Z has a pseudo equilibria on W, then Z is not chaotic on W.*

Proof. It is not difficult to see that a pseudo equilibria is neither positive nor negative minimal for Z. Therefore the proof follows straightforward from Theorem 3. □

Corollary 3. *If Z is positive (resp. negative) chaotic on W and $med(W) > 0$, then W is positive (resp. negative) minimal.*

Proof. It is enough repeat the proof of Theorem 3. □

The next result provide a sufficient condition in order to a PSVF Z be chaotic on an invariant compact set W. Additionally, it guarantee that under suitable hypotheses the periodic trajectories of Z are dense in W.

Theorem 4. *Let Z be a PSVF and W a compact positive (respect. negative) invariant set. Given $x, y \in W$, assume that there exist a positive (respect. negative) trajectory ϕ_t^+ (respect. ϕ_t^-) connecting x and y. Then Z is positive (respect. negative) chaotic on W and the positive (respect. negative) periodic trajectories of Z are dense in W.*

The last theorem is inspired in Lemma 1 and Theorems 8 and 10 of [6] and it proof is analogous to the proofs of these results, so we will not prove it here. Also, Theorem 4 leads to the next corollary:

Corollary 4. *Let Z be a PSVF and W a compact invariant set on which any two points can be connected simultaneously by positive and negative trajectories. Then Z is chaotic on W and its periodic trajectories are dense in W.*

Proof. Since every pair of points in W can be connected simultaneously by positive and negative trajectories of Z, by Theorem 4, the PSVF Z is both positive and negative chaotic on Z. So, by Theorem 1, we get that Z is chaotic on W. Moreover, since the positive and negative periodic trajectories of Z are dense in W, the density of the periodic trajectories of Z on W is straightforward. □

Acknowledgments. The first author was partially supported by grant#2014/02134-7, São Paulo Research Foundation (FAPESP), CAPES-Brazil grant number 88881.030454/2013-01 from the program CSF-PVE and a CNPq-Brazil grant number 443302/2014-6. The second author is supported by grant#2013/25828-1, São Paulo Research Foundation (FAPESP).

References

[1] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems − Theory and Applications*, Springer-Verlag (2008).

[2] M. di Bernardo, A. Colombo and E. Fossas, *Two-fold singularity in nonsmooth electrical systems*, Proc. IEEE International Symposium on Circuits ans Systems, (2011), 2713–2716.

[3] M. di Bernardo, K.H. Johansson, and F. Vasca, *Self-Oscillations and Sliding in Relay Feedback Systems: Symmetry and Bifurcations*, Internat. J. Bif. Chaos, vol. 11, (2001), pp. 1121-1140.

[4] B. Brogliato, *Nonsmooth Mechanics: Models, Dynamics and Control*, Springer- Verlag, New York, 1999.

[5] C.A. Buzzi, T. Carvalho and R.D. Euzébio, *On Poincaré-Bendixson Theorem and non-trivial minimal sets in planar nonsmooth vector fields*, submitted (arxiv link: http://arxiv.org/pdf/1307.6825v1.pdf).

[6] C.A. Buzzi, T. Carvalho and R.D. Euzébio, *Chaotic Planar Piecewise Smooth Vector Fields With Non Trivial Minimal Sets*, Ergodic Theory of Dynamical Systems, published online: http://journals.cambridge.org/abstract_S0143385714000674 (doi: 10.1017/etds.2014.67) (2014).

[7] C.A. Buzzi, T. Carvalho and M.A. Teixeira, *Birth of limit cycles bifurcating from a nonsmooth center*, Journal de Mathematiques Pures et Appliquees, vol **102**, p. 36 – 47, 2014.

[8] T. Carvalho, *On the Closing Lemma for Planar Piecewise Smooth Vector Fields*, preprint, 2014.

[9] T. Carvalho and D.J. Tonon, *Normal Forms for Codimension One Planar Piecewise Smooth Vector Fields*, International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, vol 24 (7), (2014), 1450090.

[10] A. Colombo and M.R. Jeffrey, *Nondeterministic chaos, and the two-fold singularity in piecewise smooth flows*, SIAM J. Applied Dymanical Systems, 10, vol **2**, (2011), 423–451.

[11] F. Dercole and F.D. Rossa, *Generic and Generalized Boundary Operating Points in Piecewise-Linear (discontinuous) Control Systems*, In 51st IEEE Conference on Decision and Control, 10-13 Dec. 2012, Maui, HI, USA. Pages:7714-7719. DOI: 10.1109/CDC.2012.6425950.

[12] D.D. Dixon, *Piecewise Deterministic Dynamics from the Application of Noise to Singular Equation of Motion*, J. Phys A: Math. Gen. 28, (1995), 5539-5551.

[13] R.D. Euzébio and M.R.A. Gouveia, *Poincaré Recurrence Theorem for Non-Smooth Vector Fields*, preprint, 2014.

[14] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers-Dordrecht, 1988.

[15] S. Genena, D.J. Pagano and P. Kowalczik, *Hosm Control of Stick-Slip Oscillations in Oil Well Drill-Strings*, In Proceedings of the European Control Conference 2007 - ECC07, Kos, Greece, July.

[16] A. Jacquemard and D.J. Tonon, Coupled systems of non-smooth differential equations, Bulletin des Sciences Mathématiques, vol. 136, (2012), pp. 239-255.

[17] M.R. Jeffrey, *Nondeterminism in the Limit of Nonsmooth Dynamics*, Phys. Rev. Lett, 254103 **106**, (2011) 1–4.
[18] T. Kousaka, T. Kido, T. Ueta, H. Kawakami and M. Abe, *Analysis of Border- Collision Bifurcation in a Simple Circuit*, Proceedings of the International Symposium on Circuits and Systems, II-481-II-484, (2000).
[19] R. Leine and H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Lecture Notes in Applied and Computational Mechanics, vol. 18, Berlin Heidelberg New-York, Springer-Verlag, (2004).
[20] J. Sotomayor, A.L. Machado, *Structurally stable discontinuous vector fields on the plane*, Qual. Theory Dyn. Syst., **3** (2002), 227–250.
[21] J.D. Meiss, *Differential Dynamical Systems*, SIAM, 2007.
[22] M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology, **1**, (1962) 101–120.
[23] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste: I, II, III*, Paris: Guathier-Villars, 1892, 1099, 1899. (Translated in *Izbrannye trudy* (Selected works), Moscow: Akademia Nauk, 1971.)
[24] D.J. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, World Scientific Series on Nonlinear Science, Series A, v. 69 (2010).
[25] M.A. Teixeira, *Generic Bifurcation in Manifolds with Boundary*, Journal of Differential Equations **25** (1977), 65–88.

[1] Departamento de Matemática, Faculdade de Ciências, UNESP, Av. Eng. Luiz Edmundo Carrijo Coube 14-01, Zip Code 17033-360, Bauru, SP, Brazil.
E-mail address: tcarvalho@fc.unesp.br

[2] Departamento de Matemática, IMECC–UNICAMP, Rua Sérgio Buarque de Holanda, 651, Zip code 13083–970, Campinas, São Paulo, Brazil
E-mail address: euzebio@ime.unicamp.br