ATTRACTIVITY, DEGENERACY AND CODIMENSION OF A TYPICAL SINGULARITY IN 3D PIECEWISE SMOOTH VECTOR FIELDS

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Abstract. We address the problem of understanding the dynamics around typical singular points of $3D$ piecewise smooth vector fields. A model Z_0 in $3D$ presenting a T-singularity is considered and a complete picture of its dynamics is obtained in the following way: *(i)* Z_0 has an invariant plane π_0 filled up with periodic orbits (this means that the restriction $Z_0|_{\pi_0}$ is a center around the singularity), *(ii)* All trajectories of Z_0 converge to the surface π_0, and such attraction occurs in a very non-usual and amazing way, *(iii)* given an arbitrary integer $k \geq 0$ then Z_0 can be approximated by π_0-invariant piecewise smooth vector fields Z_ε such that the restriction $Z_\varepsilon|_{\pi_0}$ has exactly k-hyperbolic limit cycles, *(iv)* the origin can be chosen as an asymptotic stable equilibrium of Z_ε when $k = 0$, and finally, *(v)* Z_0 has infinite codimension in the set of all $3D$ piecewise smooth vector fields.

1. Introduction

Our interest in this paper is to study some qualitative aspects of *piecewise smooth vector fields* (PSVFs for short) in $3D$. Roughly speaking, as stated in [5], a PSVF $Z = (X, Y)$ is a pair of C^r-vector fields X and Y (both defined on $\mathbb{R}^3$), in such a way that just their restrictions to some regions (half-spaces) separated by a codimension one surface Σ (called *switching manifold*) are considered. In Section 2 we give a precise definition.

In this context, some of the points with richer dynamics are those ones where the trajectories of X and/or Y are tangent to Σ. These points are called *tangential singularities*. The most known tangential singularity of a smooth system X is the *fold singularity* (also called *fold point*), which is characterized by the quadratic contact of an orbit of X with Σ. A fold point p can be visible or invisible. It is *visible* for X if the X-trajectory passing though p remains in the same side where X is defined, otherwise it is *invisible*. In $3D$, generically there exists a curve of tangential singularities $S_X \subset \Sigma$ passing through p (the same for Y).

If $p \in \Sigma$ is a fold singularity of both systems X and Y then it is called a *two-fold singularity*. This singularity is a prototypical model in the generic

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classification of singularities in PSVFs. As pointed out in [7], a two-fold singularity is an important organizing centre because it brings together all of the basic forms of dynamics possible in a PSVF. There are many distinct topological types of two-fold singularities and the most interesting of them is the so called *T-singularity*. We say that p is a T-singularity (or *Teixeira-singularity* – due to the pioneering work [20] – or *invisible two-fold singularity*) for $Z = (X, Y)$ if p is an invisible fold point of both X and Y and S_X meets S_Y transversally at p (see Figure 1). The interested reader can see more details about the T-singularity in [2, 3, 6, 7, 8, 12, 13, 20].

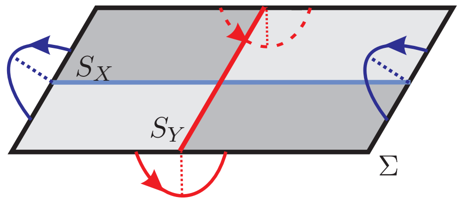

FIGURE 1. T-singularity.

Our primary concern in this article is to show the existence of a infinite codimension two-fold singularity. In order to do it, we study smooth non-linear perturbations of a specific model of $3D$ degenerate T-singularity and a complete picture of its amazing dynamics is exhibited. It is worth noting that some interesting bifurcations of such T-singularity are exhibited.

A formal codimension study of a singularity must contemplate the minimal number of parameters necessary in order to obtain all topological types of dynamical systems around the dynamical system presenting the singularity. In [5] the authors perform this study for a planar two-fold singularity of PSVFs.

Some ideas and constructions presented in [5] and [6] are adapted in our approach. As consequence, we are able to produce an arbitrary number of topological types of PSVFs in a small neighborhood of a PSVF presenting a T-singularity. So, this T-singularity has infinite codimension.

Regardless of, this work fits into a general program for understanding the dynamics of higher dimension vector fields expressed by:

$$\dot{u} = F(u) + sgn(f(u))K \tag{1}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, K is a constant n-dimensional vector, $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth mapping and $F : \mathbb{R}^n \to \mathbb{R}^n$, in general, is not smooth in the switching manifold $\Sigma = \{f^{-1}(0)\}$.

Systems like (1) are wide used in applications and appears in models of Mechanics Engineering (see [4, 10, 18]), Electric Engineering (see [2, 14]), Biologic/Control Theory/Economics (with sudden external influences, see [9]), among others. In fact, every system susceptible to *on-off* operations are modelled by systems like (1) which imposes an interdisciplinary aspect on this theory.

The *sgn function* produces, in general, lost of differentiability of (1) when the trajectories pass throught Σ. So, (1) is a non-smooth (or piecewise smooth) system. As also happens for smooth systems, some choices on the functions at the right-side of (1) generate non stable behavior. There are in the current literature a huge variety of papers dealing with stability conditions of models like (1) (see [19, 20] among others).

It is worth mentioning that in [1] Anosov proves the asymptotic stability of the origin of (1) when F is a linear vector field. After, Küpper-Kunze-Hosham found invariant varieties of (1) when $K = 0$. In fact, they show the existence of invariant varieties (cones) and prove that the trajectories of (1) twist (curl up) until it (see [15, 16, 17]).

In our approach the following $3D$ system is considered:

$$Z_0(x,y,z) = (\dot{x},\dot{y},\dot{z}) = F(x,y,z) + sgn(z)K,$$

where $f(x,y,z) = z$,

$$F(x,y,z) = \frac{1}{2}\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2}sgn(z)\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$K = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix},$$

or equivalently:

(2)
$$Z_0(x,y,z) = \begin{cases} X(x,y,z) = (-1-(x+y), 1-(x+y), -y) & \text{if } z \geq 0, \\ Y_0(x,y,z) = (1,-1,x) & \text{if } z \leq 0. \end{cases}$$

In fact, as exhibited in [16, 17], we also identify an invariant manifold in our model. Morevoer, the trajectories converge to it in a very unusual way. The richness of our model is revealed after a suitable perturbation of it. In such case we conserve the invariant manifold and identify the birth of an arbitrary number of limit cycles. Also, we obtain the asymptotic stability of the origin (as in [1]).

Now we state the main results of the paper.

Theorem A. *Let Z_0 be given by* (2) *and π_0 the plane $\{y+x=0\}$. For each integer $k \geq 0$, there exists a one-parameter family of π_0-invariant PSVFs Z_ε satisfying:*

- (a) $Z_\varepsilon \to Z_0$ when $\varepsilon \to 0$;
- (b) Z_ε has exactly k hyperbolic limit cycles in a neighborhood of the origin. The same holds for $k = \infty$ and,
- (c) All trajectories of Z_0 and Z_ε converge to π_0.
- (d) When $\varepsilon > 0$ and $k = 0$, the origin is an asymptotic stable equilibrium point for Z_ε.

An immediate consequence of Theorem A is:

Theorem B. *The PSVF Z_0, given by* (2)*, has infinite codimension.*

The paper is organized as follows. In Section 2 we introduce the terminology, some definitions and the basic theory about PSVFs. In Section 3 we present properties towards the understanding of the phase portrait of (2). In Section 4 suitable perturbations of (2) are considered and the birth of limit cycles are explicitly exhibited. In Section 5 we prove the main results. In Section 6 we made a brief conclusion about the results in this paper and in Section 7 we picture some numerical analysis and the phase portrait of (2) around the origin.

2. Preliminaries

In this section we give a brief review of the theory.

Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^3$. We consider a codimension one manifold Σ of $\mathbb{R}^3$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \,|\, f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \,|\, f(q) \leq 0\}$. Throughout paper we assume that $\Sigma = f^{-1}(0)$, where $f(x, y, z) = z$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^3$ endowed with the C^r-topology, with $r \geq 1$ large enough for our purposes. Call Ω^r the space of vector fields $Z : V \to \mathbb{R}^3$ such that

$$Z(x,y,z) = \begin{cases} X(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma^+, \\ Y(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma^-, \end{cases} \tag{3}$$

where $X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3) \in \chi$. We endow Ω^r with the product topology. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we will accept it to be multi-valued in points of Σ. The basic results of differential equations, in this context, were stated in [11].

On Σ we distinguish the following regions:

• Crossing Region: $\Sigma^c = \{p \in \Sigma \,|\, X_3(p) \cdot Y_3(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \,|\, X_3(p) > 0, Y_3(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \,|\, X_3(p) < 0, Y_3(p) < 0\}$.

• Sliding Region: $\Sigma^s = \{p \in \Sigma \,|\, X_3(p) < 0, Y_3(p) > 0\}$.

• Escaping Region: $\Sigma^e = \{p \in \Sigma \,|\, X_3(p) > 0, Y_3(p) < 0\}$.

When $q \in \Sigma^s \cup \Sigma^e$, following the Filippov's convention, the **sliding vector field** associated to $Z \in \Omega^r$ is the vector field $\widehat{Z}^s$ tangent to $\Sigma^s \cup \Sigma^e$ and expressed in coordinates as

$$\widehat{Z}^s(q) = \frac{1}{(Y_3 - X_3)(q)}((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q), 0). \tag{4}$$

Associated to (4) there exists the planar **normalized sliding vector field**

$$Z^s(q) = ((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q)). \tag{5}$$

Remark 1. *Note that, if $q \in \Sigma^s$ then $X_3(q) < 0$ and $Y_3(q) > 0$. So, $(Y_3 - X_3)(q) > 0$ and therefore, $\widehat{Z}^s$ and Z^s are topologically equivalent in Σ^s since they have the same orientation and can be C^r-extended to the closure $\overline{\Sigma^s}$ of Σ^s. If $q \in \Sigma^e$ then $\widehat{Z}^s$ and Z^s have opposite orientation.*

In this context, a rich dynamics occurs on those points $p \in \Sigma$ such that $X_3(p) \cdot Y_3(p) = 0$, called **tangential singularities of Z** (i.e., the trajectory through p is tangent to Σ).

For practical purposes, the contact between the smooth vector field X and the switching manifold $\Sigma = f^{-1}(0)$ is characterized by the expression $X.f(p) = \langle \nabla f(p), X(p)\rangle = 0$ where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^3$. In this way, we say that a point $p \in \Sigma$ is a *fold point* of X if $X.f(p) = 0$ but $X^2.f(p) \neq 0$, where $X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p)\right\rangle$ for $i \geq 2$. Moreover, $p \in \Sigma$ is a *visible* (respectively *invisible*) fold point of X if $X.f(p) = 0$ and $X^2.f(p) > 0$ (respectively $X^2.f(p) < 0$). In addition, a tangential singularity q is *singular* if q is a invisible tangency for both X and Y. On the other hand, a tangential singularity q is *regular* if it is not singular.

Call S_X (resp. S_Y) the set of all tangential singularities of X (resp. Y). In $3D$, a point of Σ at which two curves of fold singularities S_X and S_Y meet is called a *two-fold singularity*. This singularity is a prototypical model in the generic classification of singularities in PSVFs. As pointed out in [7], a two-fold singularity is an important organizing centre because it brings together all of the basic forms of dynamics possible in a PSVF. There are many distinct topological types of two-fold singularities and the most interesting of them is the so called *T-singularity*. We say that p is a T-singularity (or *Teixeira-singularity* or *invisible two-fold singularity*) for $Z = (X, Y)$ if p is an invisible fold point of both X and Y and the intersection of S_X and S_Y is transversal at p (see Figure 1). It is easy to check that in the model (2) the origin is a T-singularity. In this article, we study smooth nonlinear perturbations of this model and a complete picture of its dynamics is exhibited. The interested reader can see more details about the T-singularity in [2, 3, 6, 7, 8, 12, 13, 20].

It is worth to say that some constructions and ideas of [5] and [6] are very useful in our approach.

Now we establish a convention on the trajectories of orbit-solutions of a PSVF.

Definition 1. *The* ***local trajectory (orbit)*** $\phi_Z(t, p)$ *of a PSVF given by* (3) *through $p \in V$ is defined as follows:*

- *For $p \in \Sigma^+ \backslash \Sigma = \{q \in V \mid z > 0\}$ and $p \in \Sigma^- \backslash \Sigma = \{q \in V \mid z < 0\}$ the trajectory is given by $\phi_Z(t, p) = \phi_X(t, p)$ and $\phi_Z(t, p) = \phi_Y(t, p)$ respectively.*
- *For $p \in \Sigma^{c+}$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t, p) = \phi_Y(t, p)$ for $t \leq 0$ and $\phi_Z(t, p) = \phi_X(t, p)$ for $t \geq 0$. For the case $p \in \Sigma^{c-}$ the definition is the same reversing time.*

- *For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \geq 0$. For $p \in \Sigma^s$ the definition is the same reversing time.*
- *For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \geq 0$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*
- *For p a singular tangency point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

Definition 2. *The* ***orbit (trajectory)*** *of a point $p \in V$ is the set $\gamma(p) = \{\phi_Z(t,p) : t \in \mathbb{R}\}$ obtained by the concatenation of local trajectories.*

Consider $0 \neq p \in \Sigma^{c+}$. It is easy to see that there exists a time $t_1(p) > 0$, called X-fly time, such that the forward trajectory of X passing through p at $t = 0$ return to Σ after $t_1(p)$. We define the *half return map associated to* X by $\varphi_X(p) = \phi_X(t_1(p), p) = p_1 \in \Sigma$. When $p_1 \in \Sigma^{c-}$, let $t_2(p_1) > 0$ be the Y-fly time of the trajectory of Y passing through p_1. Define the *half return map associated to* Y by $\varphi_Y(p_1) = \phi_Y(t_2(p_1), p_1) \in \Sigma$. The C^r involution φ_X (resp. φ_Y) is such that $Fix(\varphi_X) = S_X$ (resp. $Fix(\varphi_Y) = S_Y$). The *first return map* associated to $Z = (X, Y)$ is defined by the composition of these involutions, i.e.,

$$\varphi_Z(p) = \varphi_Y \circ \varphi_X(p) = \phi_Y(t_2(p_1), \phi_X(t_1(p), p)) \tag{6}$$

or the reverse, applying first the flow of Y and after the flow of X. See Figure 2 and details in [21].

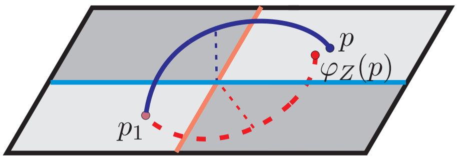

Figure 2. First Return Map.

The mapping φ_Z is an important object in order to study the behavior of Z around a T-singularity. The proofs of the main results require a detailed analysis of the return map and different domains (departure regions), according to the Filippov's decomposition of Σ, must be considered. For example, if $q \in \Sigma^e$ then the iteration of both mappings $\varphi_Y \circ \varphi_X(q)$ and $\varphi_X \circ \varphi_Y(q)$ must be considered.

3. Properties of System Z_0 given by (2)

In this section we describe some important features about the PSVF given by (2). In Subsection 3.1 we describe the sliding vector field associated to (2). In Subsection 3.2 we analyze the first return map associated to it. In Subsection 3.3 we analyze the way in which the trajectories converge to a limit set. This last analysis permits us to detect the behavior of certain invariant sets.

3.1. **The sliding vector field associated to** (2)**.** Using Equation (5), associated to (2) we have the normalized sliding vector field

$$Z_0^s(x,y,z) = ((-1-(x+y))x + y, (1-(x+y))x - y, 0). \tag{7}$$

Let us understand the phase portrait of (7).

Proposition 3. *The normalized sliding vector field Z_0^s, given by* (7)*, has a saddle-node at the origin.*

Proof. Identify Σ with the xy-plane. Consider the change of variables $(u,v) = (x+y, x-y)$. Then (7) can be re-written in the form $(\dot{u},\dot{v}) = (-(u+v)u, -2v)$. This last system has the origin as a unique equilibrium with eigenvectors $v_1 = (1,0)$, $v_2 = (0,1)$ associated to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = -2$ respectively. The phase portrait is pictured at Figure 3.

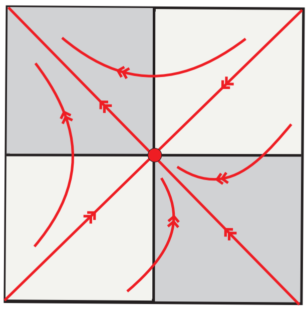

FIGURE 3. Phase Portrait of the normalized sliding vector field Z_0^s.

□

Remark 2. *Since* (7) *has a saddle-node at the origin, by Remark 1 we must reverse the orientation in Σ^e. So, we conclude that the sliding vector field associated to Z_0 has the phase portrait shown at Figure 4. Note that the straight line $y+x=0$ in Σ, where $Y_3 - X_3 = 0$ in* (4)*, is composed only by equilibrium points of the sliding vector field associated to Z_0. Moreover, except by the stable invariant manifold, the trajectories of the sliding vector field associated to Z_0 depart from Σ^e.*

3.2. **The first return map associated to** (2)**.** In order to exhibit the first return map associated to $Z_0 = (X, Y_0)$, given by (2), we have to write the expressions of the half return maps φ_X and φ_{Y_0}. A straightforward calculation shows that the trajectories ϕ_X and ϕ_{Y_0} are parameterized, respectively, by

$$\phi_X(t) = (-t + k_1 e^{-2t} + k_2, t + k_1 e^{-2t} - k_2, -t^2/2 + k_1 e^{-2t}/2 + k_2 t + k_3), \tag{8}$$

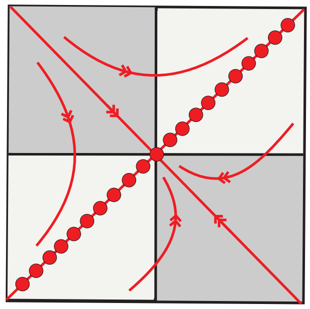

FIGURE 4. Phase Portrait of the sliding vector field associated to Z_0.

and

$$\phi_{Y_0}(t) = (t + l_1, -t + l_2, t^2/2 + l_1 t + l_1^2/2 + l_3), \tag{9}$$

Consider the planes $\pi_k = \{(x, y, z) \in V \mid y = -x + k\}$, with $k \in \mathbb{R}$.

Proposition 4. *Given an arbitrary point* $(x_0, y_0, 0) \in \Sigma^{c+}$ *then* $\varphi_X(x_0, y_0, 0) = (-t_2 + ((x_0+y_0)/2)e^{-2t_2} + (x_0-y_0)/2, t_2 + ((x_0+y_0)/2)e^{-2t_2} - (x_0-y_0)/2, 0)$*, where the* X*-fly time* $t_2 > 0$ *is given implicitly by*

$$-t_2^2/2 + ((x_0 + y_0)/4)e^{-2t_2} + ((x_0 - y_0)/2)t_2 - (x_0 + y_0)/4 = 0. \tag{10}$$

In particular, $\phi_X(\pi_0 \cap \Sigma^+) \subset (\pi_0 \cap \Sigma^+)$ *and* $\varphi_X(x_0, -x_0, 0) = (-x_0, x_0, 0)$.

Proof. Considering the initial condition $(x_0, y_0, 0)$ and (8), let $t_2 > 0$ be the first time such that

$$-t^2/2 + ((x_0 + y_0)/4)e^{-2t} + ((x_0 - y_0)/2)t - (x_0 + y_0)/4 = 0.$$

Using (8) it is easy to see that

$$\varphi_X(x_0, y_0, 0) =$$
$$(-t_2 + ((x_0+y_0)/2)e^{-2t_2} + (x_0-y_0)/2, t_2 + ((x_0+y_0)/2)e^{-2t_2} - (x_0-y_0)/2, 0).$$

In particular, for $p_0 = (x_0, y_0, 0) = (x_0, -x_0, 0) \in \pi_0 \cap \Sigma$, we obtain that the first two coordinates of $\phi_X(t, p_0)$ are $x(t) = -t + x_0$ and $y(t) = t - x_0 = -x(t)$. Moreover, in this case we can solve explicitly the equation $-t_2^2/2 + ((x_0 + y_0)/2)e^{-2t_2}/2 + ((x_0 - y_0)/2)t_2 - (x_0 + y_0)/4 = 0$ and we obtain $t_2 = 2x_0$. So, $\varphi_X(x_0, -x_0, 0) = (-x_0, x_0, 0)$. This concludes the proof. □

Proposition 5. *Given an arbitrary point* $(x_0, y_0, 0) \in \Sigma^{c-}$ *then* $\varphi_{Y_0}(x_0, y_0, 0) = (-x_0, y_0 + 2x_0, 0)$*. In particular,* $\phi_{Y_0}(\pi_k \cap \Sigma^-) \subset (\pi_k \cap \Sigma^-)$.

Proof. Considering the initial condition $(x_0, y_0, 0)$ and (9), in order to determine φ_{Y_0} it is enough to obtain the first time $t_1 > 0$ such that $t_1^2/2 + x_0 t_1 = 0$. So $t_1 = -2x_0$ and $\varphi_{Y_0}(x_0, y_0, 0) = (-x_0, y_0 + 2x_0, 0)$. In particular, for $p_0 = (x_0, y_0, 0) = (x_0, -x_0 + k, 0) \in \pi_k \cap \Sigma^-$, the first two coordinates of $\phi_Y(t, p_0)$ are $x(t) = t + x_0$ and $y(t) = -t - x_0 + k = -(t + x_0) + k = -x(t) + k$. This concludes the proof. □

Proposition 6. *The plane* $\pi_0 = \{(x, y, z) \in V \,|\, x + y = 0\}$ *is* Z_0*-invariant. Moreover,* $Z_0|_{\pi_0}$ *is a center.*

Proof. By Propositions 4 and 5 we get that π_0 is invariant by the flow of Z_0. In order to see that Z_0 has a center at π_0 it is enough to see that

$$\varphi_{Z_0}(x_0, -x_0, 0) = \varphi_{Y_0} \circ \varphi_X(x_0, -x_0, 0) = \varphi_{Y_0}(-x_0, x_0, 0) = (x_0, -x_0, 0).$$

□

Now we will prove that π_0 is a hyperbolic global attractor for the trajectories of Z_0.

Proposition 7. *Let* r_0 *be the straight line given by* $r_0 = \pi_0 \cap \Sigma$*. Given* $(x_0, y_0, 0) \in \Sigma^c$ *then*

$$d(\varphi_{Z_0}(x_0, y_0, 0), r_0) < d((x_0, y_0, 0), r_0).$$

This means that the trajectories of Z_0 *are converging to* r_0*. Moreover,*

$$\varphi^n_{Z_0}(x_0, y_0, 0) = (-x_n, x_n + (x_0 + y_0)e^{-2(t_2^{(1)}+\ldots+t_2^{(n)})}, 0)$$

where $t_2^{(i)}$ *is the fly time necessary to the* X*-trajectory by* $\varphi^i_{Z_0}(x_0, y_0, 0)$ *returns to* Σ *and* $x_n = -t_2^{(n)} + ((x_0 + y_0)/2)e^{-2(t_2^{(1)}+\ldots+t_2^{(n)})} - x_{n-1} - (x_0 + y_0)(e^{-2(t_2^{(1)}+\ldots+t_2^{(n-1)})})/2$*.*

Proof. By Proposition 4 we obtain

$$\varphi_X(x_0, y_0, 0) =$$
$$(-t_2+((x_0+y_0)/2)e^{-2t_2}+(x_0-y_0)/2, t_2+((x_0+y_0)/2)e^{-2t_2}-(x_0-y_0)/2, 0),$$

where $t_2 > 0$ is given implicitly by

$$-t_2^2/2 + ((x_0 + y_0)/2)e^{-2t_2}/2 + ((x_0 - y_0)/2)t_2 - (x_0 + y_0)/4 = 0.$$

By Proposition 5,

(11)
$$\varphi_{Z_0}(x_0, y_0, 0) = (\varphi_Y \circ \varphi_X)(x_0, y_0, 0) =$$
$$= \Big(t_2 - (\tfrac{(x_0+y_0)}{2}e^{-2t_2} - \tfrac{(x_0-y_0)}{2}, -t_2 + 3(\tfrac{(x_0+y_0)}{2}e^{-2t_2} + \tfrac{(x_0-y_0)}{2}, 0\Big).$$

Then we get,

$$d(\varphi_{Z_0}(x_0, y_0, 0), r_0) = \frac{\sqrt{2}}{2}(x_0 + y_0)e^{-2t_2} < \frac{\sqrt{2}}{2}(x_0 + y_0) = d((x_0, y_0, 0), r_0).$$

In order to obtain that

$$\varphi^n_{Z_0}(x_0, y_0, 0) = (-x_n, x_n + (x_0 + y_0)e^{-2(t_2^{(1)}+\ldots+t_2^{(n)})}, 0),$$

with

$$x_n = -t_2^{(n)} + \frac{(x_0 + y_0)}{2}e^{-2(t_2^{(1)}+\ldots+t_2^{(n)})} - x_{n-1} - \frac{(x_0 + y_0)(e^{-2(t_2^{(1)}+\ldots+t_2^{(n-1)})})}{2},$$

it is enough to use n times Propositions 4 and 5. □

Proposition 8. *Let* $p_0 = (x_0, y_0, 0) \in \pi_0$. *The first return map* $\varphi_Z(p_0)$ *has eigenvalues* $\mu_1 = 1$ *and* $\mu_2 = 1 - 4x_0 + 8x_0^2 + O(x_0^2)$ *and eigenvectors* $u_1 = (-1, 1)$ *and* $u_2 = (-(1 - 2x_0 + 2x_0)/(1 - 4x_0 + 6x_0^2) + O(x_0^2), 1)$, *respectively.*

Proof. Since we can not solve Equation (10) we consider the second order series expansion of e^{-2t}. So we are able to solve (10) and obtain the X-fly time of p_0. Moreover, a straightforward calculation shows that the diffeomorphism $\varphi_Z(p_0)$ has eigenvalues $\mu_1 = 1$ and $\mu_2 = 1 - 4x_0 + 8x_0^2$ and eigenvectors $u_1 = (-1, 1)$ and $u_2 = (-(1 - 2x_0 + 2x_0)/(1 - 4x_0 + 6x_0^2), 1)$, respectively. □

Proposition 9. *All trajectories of* Z_0, *given by* (2), *converge to the plane* π_0.

Proof. First note that each $q \in V$ hits Σ for some positive time. Using Remark 2, the position of the eigenspaces of (7) and Propositions 7 and 8 it is easy to see that given $p \in \Sigma = \Sigma^s \cup \Sigma^d \cup \Sigma^c$ the trajectories of Z_0 by p converge to π_0. □

3.3. **The convergence of the trajectories.** Now we picture the scenario describing the asymptotic behavior of Z_0 in V. As we said above, the plane π_0 is an attractor for $\phi_Z(t, p)$ and separates V in two open regions $\mathcal{V}^+ = \{(x, y, z) \in V \,|\, x + y > 0\}$ and $\mathcal{V}^- = \{(x, y, z) \in V \,|\, x + y < 0\}$.

Since all points of V hit Σ, in order to determine the asymptotic behavior of a trajectory it is enough to take this trajectory departing just from points in Σ. Consider the partition of $\Sigma = \Sigma_s^{c+} \cup \Sigma_e^{c+} \cup \Sigma^e \cup \Sigma_e^{c-} \cup \Sigma_s^{c-} \cup \Sigma^s \cup r_0^+ \cup r_0^- \cup S_Y^- \cup S_X^- \cup S_Y^+ \cup S_X^+ \cup 0\}$, where $\Sigma_s^{c+} = \Sigma^{c+} \cap V^+$, $\Sigma_e^{c+} = \Sigma^{c+} \cap V^-$, $\Sigma_e^{c-} = \Sigma^{c-} \cap V^-$, $\Sigma_s^{c-} = \Sigma^{c-} \cap V^+$, $r_0^+ = r_0 \cap \Sigma^{c+}$, $r_0^- = r_0 \cap \Sigma^{c-}$, $S_Y^- = S_Y \cap V^-$, $S_X^- = S_X \cap V^-$, $S_Y^+ = S_Y \cap V^+$, $S_X^+ = S_X \cap V^+$ and $0 = \{(0, 0, 0)\}$. See Figure 5.

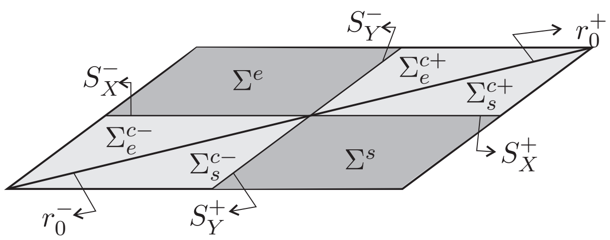

FIGURE 5. Partition of Σ.

Consider the following steps:

(1) First let us analyze the trajectories in r_0.

(1.i) If $p = 0$ then, according to the fifth bullet of Definition 1, $\phi_Z(t, p) = 0$ for all $t \in \mathbb{R}$.

(1.ii) If $p \in r_0^- \cup r_0^+$ then, according to Proposition 6, $\phi_Z(p)$ became restrict to π_0 and describes a periodic orbit around the origin.

(2) Let us analyze the trajectories in $V^+ \cap \Sigma$.

(2.i) If $p \in \Sigma^s$ then, according to Proposition 3, $\phi_Z(t,p) \to 0$ when $t \to +\infty$. By the fourth bullet of Definition 1, the same holds when $p \in S_Y^+ \cup S_X^+$.

(2.ii) If $p = (x_p, y_p, 0) \in \Sigma_s^{c+}$ then we can put $y_p = -Mx_p$, where $0 < M < 1$. With this, we are able to explicitly calculate the successive returns $\varphi_Z^n(p) = (\Pi_1^n(p), \Pi_2^n(p), 0)$, with $n \geq 1$. A straightforward calculation shows that $\Pi_2^n(0)$ is positive when $2n/(2n+1) > M$. This implies that after the first integer N such that $2N/(2N+1) > M$ we obtain $\Pi_2^n(p) > 0$ and so $\Pi^n(p)$ belongs to Σ^s, where $y > 0$. By the analysis done in (2.i), $\phi_Z(t,p) \to 0$ when $t \to +\infty$. This means that the competition between the two attractors π_0 and Σ^s is won by Σ^s.

(2.iii) If $p = (x_p, y_p, 0) \in \Sigma_s^{c-}$ then $\phi_Z(p, -2x_p) \in \Sigma_s^{c+}$ and so, repeat the analysis of item (2.ii).

(3) Let us analyze the trajectories in $V^- \cap \Sigma$.

(3.i) If $p \in \Sigma_e^{c+} \cup S_Y^-$ then we can put $y_p = -Rx_p$, where $0 < R < 1$. With this, we are able to explicitly calculate the successive returns $\varphi_Z^n(p) = (\Pi_1^n(p), \Pi_2^n(p), 0)$, with $n \geq 1$. A straightforward calculation shows that $\varphi_Z^n(p) \to \infty$. So, $\phi_Z(t,p) \to \infty$ when $t \to +\infty$.

(3.ii) If $p \in \Sigma_e^{c-} \cup S_X^-$ then $\phi_Z(p, -2x_p) \in \Sigma_e^{c+}$ and so, repeat the analysis of the previous bullet.

(3.iii) If $p \in \Sigma^e$ then, according to the third bullet of Definition 1, there are three choices for $\phi_Z(t,p)$. When $\phi_Z(t,p) = \phi_X(t,p)$ or when $\phi_Z(p,t) = \phi_Y(t,p)$ the trajectory hits Σ_e^{c-} or Σ_e^{c+}, respectively. In both cases we use the previous two items. When $\phi_Z(t) = \phi_{Z^s}(t,p)$ then, by Remark 2, there exists an invariant manifold in Σ^e converging to the origin and the other points in Σ^e converge to $S_Y^- \cup S_X^-$ (and so, we use the previous two items). This means that the competition between the attractor π_0 and the repulsiveness of Z in $p \in \Sigma_e^{c+}$ is won by the second one.

Remark 3. *As consequence of the previous analysis we are able to say that any Z-trajectory, with Z given by (2), through $p \in V$ either describes a periodic orbit or converges to a stationary point or leaves any ball around the origin, according to the initial position of p.*

4. Properties of an oriented perturbation of System Z_0

4.1. **Auxiliary results.** In what follows, $h:\mathbb{R}\rightarrow\mathbb{R}$ will denote the C^{∞}-function given by

$$h(w)=\begin{cases} 0, & \text{if } w\leq 0;\\ e^{-1/w}, & \text{if } w>0.\end{cases}$$

Consider the function $F_{\varepsilon}^{\rho}(x,y):\mathbb{R}^2\times\mathbb{R}$, where either $\rho=f$ or $\rho=i$, such that

$$F_{\varepsilon}^{f}(x,y)=-\varepsilon h(x)h(-y)(\varepsilon-x)(2\varepsilon-x)\dots(k\varepsilon-x) \tag{12}$$

with $k\in\mathbb{N}$, and

$$F_{\varepsilon}^{i}(x,y)=h(x)h(-y)\sin(\pi\varepsilon^2/x). \tag{13}$$

Lemma 10. *Consider the function $F_{\varepsilon}^{f}(x,y)$ given by* (12)*.*

(i) *If $\varepsilon<0$ then F_{ε}^{f} does not have roots in $(0,+\infty)\times\{y\}$.*
(ii) *If $\varepsilon>0$ then F_{ε}^{f} has exactly k roots in $(0,+\infty)\times\{y\}$, these roots are $\{(\varepsilon,y),(2\varepsilon,y),\dots,(k\varepsilon,y)\}$.*
(iii) $\dfrac{\partial F_{\varepsilon}^{f}}{\partial x}(j\varepsilon,y)=-\epsilon h(-y)(-1)^j\varepsilon^k h(j\varepsilon)(k-j)!(j-1)!$ *for $j\in\{1,2,\dots,k\}$. It means that such partial derivative at $(j\varepsilon,y)$ is positive for j odd and negative for j even.*

Proof. When $x>0$, by a straightforward calculation $F_{\varepsilon}^{f}(x,y)=0$ if, and only if, $(\varepsilon-x)(2\varepsilon-x)\dots(k\varepsilon-x)=0$. So, the roots of $F_{\varepsilon}^{f}(x,y)$ in $(0,+\infty)$ are $\varepsilon,2\varepsilon,\dots,k\varepsilon$. Moreover,

$$\frac{\partial F_{\varepsilon}^{f}}{\partial x}(x,y)=-\epsilon h(-y)\frac{\partial}{\partial x}\Big((j\varepsilon-x)H(x)\Big)=-\epsilon h(-y)\Big((j\varepsilon-x)\frac{\partial H}{\partial x}(x)-H(x)\Big),$$

where $H(x)=F_{\varepsilon}^{f}(x,y)/(\epsilon h(-y)(j\varepsilon-x))$. So,

$$\begin{aligned}&\frac{\partial F_{\varepsilon}^{f}}{\partial x}(j\varepsilon,y)=\epsilon h(-y)H(j\varepsilon)=\\ &=\epsilon h(-y)\varepsilon^k h(j\varepsilon)(1-j)\dots((j-1)-j)((j+1)-j)\dots(k-j)\\ &=-\epsilon h(-y)\varepsilon^k h(j\varepsilon)(-1)^j\Big((j-1)\dots(j-(j-1))\Big)\Big(((j+1)-j)\dots(k-j)\Big)\\ &=-\epsilon h(-y)(-1)^j\varepsilon^k h(j\varepsilon)(k-j)!(j-1)!\end{aligned}$$

This proves items (ii) and (iii). Item (i) follows immediately. □

Lemma 11. *Consider the function $F_{\varepsilon}^{i}(x,y)$ given by* (13)*. For $\varepsilon\neq 0$ the function F_{ε}^{i} has infinitely many roots in $(0,\varepsilon^2)\times\{y\}$, these roots are $\{(\varepsilon^2,y),(\varepsilon^2/2,y),(\varepsilon^2/3,y),\dots\}$ and*

$$\frac{\partial F_{\varepsilon}^{i}}{\partial x}(\varepsilon^2/j,y)=h(-y)(-1)^j(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j)\ \textit{for}\ j\in\{1,2,3,\dots\}.$$

It means that such derivative at $(\varepsilon^2/j,y)$ is positive for j odd and negative for j even.

Proof. When $x > 0$, by a straightforward calculation $F_\varepsilon^i(x,y) = 0$ if, and only if, $\sin(\pi\varepsilon^2/x) = 0$. So, the roots of $F_\varepsilon^i(x,y)$ in $(0,\varepsilon^2) \times \{y\}$ are $(\varepsilon^2, y), (\varepsilon^2/2, y), (\varepsilon^2/3, y), \ldots$. Moreover,

$$\frac{\partial F_\varepsilon^i}{\partial x}(x,y) = h(-y)[h'(x)\sin(\pi\varepsilon^2/x) + h(x)\cos(\pi\varepsilon^2/x)(-\pi\varepsilon^2/x^2)].$$

So,

$$\begin{aligned}\frac{\partial F_\varepsilon^i}{\partial x}(\varepsilon^2/j, y) &= h(-y)[h'(\varepsilon^2/j)\sin(\pi\varepsilon^2/j)+\\ &+ h(\varepsilon^2/j)\cos(\pi\varepsilon^2/j)(-\pi j^2/\varepsilon^2)]\\ &= h(-y)(-1)^j(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j).\end{aligned}$$

□

Consider Z_0 given by (2) and

(14)
$$Z_\varepsilon^\rho(x,y,z) = \begin{cases} X(x,y,z) = (-1-(x+y), 1-(x+y), -y) & \text{if } z \geq 0,\\ Y_\varepsilon^\rho(x,y,z) = (1,-1,x+\frac{\partial F_\varepsilon^\rho}{\partial x}(x,y)) & \text{if } z \leq 0.\end{cases}$$

Remark 4. *Take $Z_\varepsilon = Z_\varepsilon^\rho$, with $\rho = i, f$. It is easy to see that $Z_\varepsilon \to Z_0$ when $\varepsilon \to 0$.*

Associated to (14) we have the normalized sliding vector field given by

$$Z_{\rho,\varepsilon}^s(x,y,z) =$$

$$\Big((-1-(x+y))(x+\frac{\partial F_\varepsilon^\rho}{\partial x}(x,y)) + y, (1-(x+y))(x+\frac{\partial F_\varepsilon^\rho}{\partial x}(x,y)) - y, 0\Big).$$

A straightforward calculation shows that the trajectory $\phi_{Y_\varepsilon^\rho}$ of Y_ε^ρ given in (14) are parameterized by

$$\phi_{Y_\varepsilon^\rho}(t) = (t+l_1, -t+l_2, t^2/2 + l_1 t + F_\varepsilon^\rho(t+l_1, -t+l_2) + l_3). \tag{15}$$

Proposition 12. *Given an arbitrary point $(x_0,y_0,0) \in \Sigma^{c-}$ then $\varphi_{Y_\varepsilon^\rho}(x_0,y_0,0) = (t_1+x_0, -t_1+y_0, 0)$, where the Y_ε^ρ-fly time $t_1 > 0$ is given implicitly by $t_1^2/2 + x_0t_1 + F_\varepsilon^\rho(t_1+x_0, -t_1+y_0) = 0$. In particular, $\phi_{Y_\varepsilon^\rho}(\pi_k \cap \Sigma^-) \subset (\pi_k \cap \Sigma^-)$.*

Proof. Let $t_1 > 0$ the first time such that $t_1^2/2 + x_0t_1 + F_\varepsilon^\rho(t_1+x_0, -t_1+y_0) = 0$. Using (15) it is easy to see that $\varphi_{Y_\varepsilon^\rho}(x_0,y_0,0) = (t_1+x_0, -t_1+y_0, 0)$. In particular, for $p_0 = (x_0,y_0,0) = (x_0, -x_0+k, 0) \in \pi_k \cap \Sigma^-$, the first two coordinates of $\phi_{Y_\varepsilon^\rho}(t,p_0)$ are $x(t) = t+x_0$ and $y(t) = -t-x_0+k = -(t+x_0)+k = -x(t)+k$. This concludes the proof. □

Proposition 13. *The plane π_0 is invariant by the flow of Z_ε^ρ, given by (14).*

Proof. By Proposition 12 and Proposition 4 we get that π_0 is invariant by the flow of Z_ε^ρ. □

Remark 5. *Since, by Proposition 7, we get that all trajectories of Z_0 converge to π_0 we obtain that, for ε sufficiently small, the same holds for the trajectories of Z_ε^ρ.*

Proposition 14. *Consider an integer $k \geq 0$ and Z_ε^ρ given by* (14)*, where either $\rho = f$ or $\rho = i$. Then Z_ε^f has exactly k limit cycles and Z_ε^i has infinite many limit cycles, all of then situated in π_0.*

Proof. According to Remark 5, π_0 is a global attractor for Z_ε^ρ. Also, according to Proposition 13, π_0 is invariant by the flow of Z_ε^ρ. So, if there exists limit cycles, then they are situated at π_0. Moreover when we restrict the flow of Z_ε^ρ to π_0, by Propositions 12 and 4, the fixed points of the first return map $\varphi_{Z_\varepsilon^\rho} = \varphi_{Y_\varepsilon^\rho} \circ \varphi_X$ occurs when $t = t_3 = 4x_0$. So, take $p_0 = (x_0, -x_0, 0)$ and we get

$$\varphi_{Z_\varepsilon^\rho}(p_0) = \phi_{Y_\varepsilon^\rho}(2x_0, -p_0) = (x_0, -x_0, F_\varepsilon^\rho(x_0, -x_0)). \tag{16}$$

When $\rho = f$, by Item (ii) of Lemma 10,

$$\varphi_{Z_\varepsilon^\rho}(x_0, -x_0, 0) = (x_0, -x_0, 0) \Leftrightarrow x_0 = j\varepsilon \text{ with } \varepsilon > 0 \text{ and } j = 1, 2, \ldots, k.$$

Therefore, Z_ε^f has k limit cycles, all of then situated in π_0. When $\rho = i$, by Lemma 11,

$$\varphi_{Z_\varepsilon^\rho}(x_0, -x_0, 0) = (x_0, -x_0, 0) \Leftrightarrow x_0 = \varepsilon^2/j \text{ with } j = 1, 2, \ldots.$$

Therefore, Z_ε^i has infinite many limit cycles, all of then situated in π_0. □

Proposition 15. *All limit cycles in Proposition 14 are hyperbolic. Moreover, for $\epsilon > 0$, if $j = even$ then it is attractor and if $j = odd$ then it is repeller.*

Proof. In fact, in order to prove this we must consider the expression of the derivatives in Item (iii) of Lemma 10 and Lemma 11. Observe that when $k = 0$ in Proposition 14 there is not limit cycles. In this Item (iii) of Lemma 10 also is true and the origin is an attractor equilibrium of the system. □

Remark 6. *Observe that when $k = 0$ in Proposition 14 the system has not limit cycles. In this Item (iii) of Lemma 10 also is true and the origin is an attractor equilibrium of the system.*

4.2. **About the convergence of the trajectories of Z_ε^ρ, given by** (14)**.** Now we will proceed the analysis of the behavior of the trajectories of (14) in a neighborhood of the T-singularity (at the origin).

As stated in Propositions 13 and Remark 5, π_0 is invariant by the flow of (14) and all trajectories of (14) in a neighborhood of the T-singularity converge to it. However, after the perturbation imposed to Z_0, the asymptotic behavior of the trajectories can drastically changes. The biggest change occurs in V^- when $\rho = f$ and $\epsilon > 0$. Following Proposition 15, we can separate the analysis of this situation, essentially, in three cases:

- When $k = 0$ we get that Z_ε^ρ does not presents limit cycles. So, the trajectories in $V^- \backslash \Sigma^e$ becomes increasingly distant from the origin until certain moment. However, since the origin is an attractor when we restrict the analysis to π_0, there exists a moment from which the trajectory starts to converge to origin. So, the T-singularity at the origin of (14) is asymptotically stable.

• When $k \neq 0$ is even we get that Z^{ρ}_{ε} presents k hyperbolic **nested** limit cycles and Γ_k, the bigger of them, is attractor. So, the trajectories in $V^{-}\backslash\Sigma^{e}$ becomes increasingly distant from the origin until certain moment. However, since the bigger limit cycle is an attractor when we restrict the analysis to π_0, there exists a moment from which the trajectory starts to converge to it. The same holds for all attractor limit cycles at the interior of Γ_k. So, the T-singularity at the origin of (14) is asymptotically stable. Also, each one of the repeller hyperbolic limit cycles at the interior of Γ_k is repeller and each one of the attractor hyperbolic limit cycles is an attractor for the trajectories in $V^{-} \cup \pi_0$. In fact, there are topological half-cylinder of orbits converging to each it one of them.

• When $k \neq 0$ is odd we get that Z^{ρ}_{ε} presents k hyperbolic **nested** limit cycles and Γ_k, the bigger of them, is repeller. So, there are trajectories in $V^{-}\backslash\Sigma^{e}$ leaving any neighborhood of the origin. Also, there are trajectories in $V^{-}\backslash\Sigma^{e}$ converging to the origin or, when $k \geq 3$, to an inward hyperbolic limit cycle. Moreover, the T-singularity at the origin of (14) is asymptotically stable. Each one of the repeller hyperbolic limit cycles is a repeller and each one of the attractor hyperbolic limit cycles is an attractor for the trajectories in $V^{-} \cup \pi_0$. In fact, there are topological half-cylinder of orbits converging to each it one of them.

5. Proof of main results

Now we prove the main results of the paper:

***Proof of Theorem A:*.** Item (a): It follows from Remark 4.

Item (b): It follows from Propositions 14 and 15.

Item (c): It follows from Propositions 9 and Remark 5.

Item (d): It follows from Remark 6.

□

Before to prove Theorem B, let us define the classical notion of codimension of vector fields.

Definition 16. *Consider $\Theta(W) \subset \Omega^r$ a small neighborhood a vector field W. We say that W has codimension k if it appears k distinct topological types of vector fields in $\Theta(W)$.*

***Proof of Theorem B:*.** Suppose that the codimension of the T-singularity in 2 is $m < \infty$. Then, in a neighborhood of Z_0 there are PSVFs of, at most, m distinct topological types. This is a contradiction due Theorem A. So, the codimension of this singularity is infinite. □

6. Conclusion

As usual, the main aim of the perturbation theory is to approximate a given dynamical system by a more familiar one, regarding the former as a perturbation of the latter. The problem is to deduce dynamical properties of the *unperturbed* from the *perturbed* case. In this sense, we focus on certain PSVFs Z_ε^ρ which are deformations of Z_0.

In what follows we present a rough description of some properties that $Z_0 = (X, Y_0)$ and $Z_\varepsilon^\rho = (X, Y_\varepsilon^\rho)$ enjoy simultaneously. Some of them are:

- The origin is an equilibrium point of both $Z_0 = (X, Y_0)$ and $Z_\varepsilon^\rho = (X, Y_\varepsilon^\rho)$. Moreover, the tangency sets S_X and S_{Y_0} (resp. $S_{Y_\varepsilon^\rho}$), meet transversally at the origin,
- The plane $\pi_0 = \{y + x = 0\}$ is Z_0 and Z_ε^ρ-invariant.
- The sliding vector fields associated to Z_0 and Z_ε^ρ are topologically equivalent.

Also, there are properties that Z_0 and Z_ε^ρ do not enjoy simultaneously. For example:

- The T-singularity at the origin of Z_ε^ρ is asymptotically stable.
- The PSVF Z_0 restricted to π_0 has a center while in Z_ε^ρ this center is perturbed giving rise to hyperbolic limit cycles at π_0.

Finally, as a conclusion of this paper we are able to say that the T-singularity of the PSVF (2) has infinite codimension.

7. Appendix

Now we illustrate the theoretical analysis performed by means of some numerical simulations. In the next illustrations we use the computer program entitled "Mathematica".

We use the following line of commands:

$$\begin{gathered} solutions = Table[First[NDSolve[\{x'[t] == If[z[t] > 0, \\ -1 - (x[t] + y[t]), 1], y'[t] == If[z[t] > 0, 1 - (x[t] + y[t]), -1], \\ z'[t] == If[z[t] > 0, -y[t], x[t]], x[0] == \theta, y[0] == -\theta, \\ z[0] == 0\}, \{x, y, z\}, \{t, 0, 2.5\}]], \{\theta, 0.1, 2\pi - 0.1, 0.1\}]; \end{gathered}$$

$$\begin{gathered} solutions2 = Table[First[NDSolve[\{x'[t] == If[z[t] > 0, \\ -1 - (x[t] + y[t]), 1], y'[t] == If[z[t] > 0, 1 - (x[t] + y[t]), -1], \\ z'[t] == If[z[t] > 0, -y[t], x[t]], x[0] == \cos[\theta]/4, y[0] == \sin[\theta]/4, \\ z[0] == 0\}, \{x, y, z\}, \{t, 0, 2.5\}]], \{\theta, 0.1, 2\pi - 0.1, 0.1\}]; \end{gathered}$$

$$\begin{gathered} c1 = ContourPlot3D[z, \{x, -.4, .4\}, \{y, -.4, .4\}, \{z, -.5, .5\}, \\ Contours \to 0, Mesh \to False]; \end{gathered}$$

$$c2 = ContourPlot3D[y, \{x, -.5, .5\}, \{y, -.4, .4\}, \{z, -.5, .5\},$$

$$Contours \to 0, Mesh \to False];$$

$$c3 = ContourPlot3D[x + y, \{x, -.5, .5\}, \{y, -.4, .4\}, \{z, -.5, .5\},$$
$$Contours \to 0, Mesh \to False];$$

The command

$$p3 = ParametricPlot3D[Evaluate[\{x[t], y[t], z[t]\}/.solutions], \{t, 0, 2.5\},$$
$$PlotStyle \to \{Thickness[.0015], Red\}]; Show[p3, ImageSize \to Large]$$

generates Figure 6.

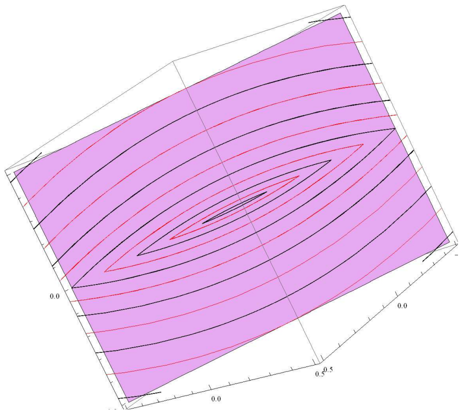

FIGURE 6. Phase Portrait of the center of Z_0.

The command

$$p4 = ParametricPlot3D[Evaluate[\{x[t], y[t], z[t]\}/.solutions2], \{t, 0, 2.5\},$$
$$PlotStyle \to \{Thickness[.0015], Red\}]; Show[p4, ImageSize \to Large]$$

generates Figure 7.

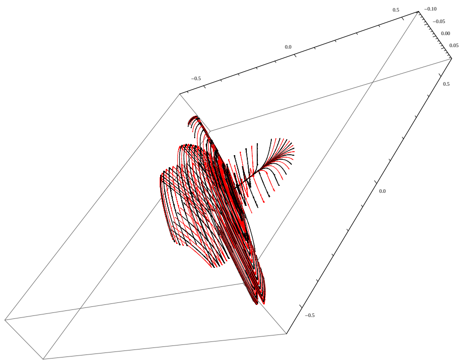

FIGURE 7. Phase Portrait of trajectories in a neighborhood of the origin.

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